Stochastic feedback, nonlinear families of Markov processes, and nonlinear Fokker-Planck equations

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Abstract

We discuss two fundamental aspects of Fokker-Planck equations that are nonlinear with respect to probability densities. First, we show that evolution equations of this kind describe processes involving stochastic feedback and interpret stochastic feedback processes in terms of hitchhiker processes and path integral solutions. Second, we demonstrate that nonlinear Fokker-Planck equations can be interpreted as linear Fokker-Planck equations describing nonlinear families of Markov diffusion processes. We exploit this finding in order to derive complete hierarchies of probability densities from nonlinear Fokker-Planck equations.

Key words: stochastic feedback; linear and nonlinear families of Markov processes; nonlinear Fokker-Planck equations

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1 Introduction

Nonlinear Fokker-Planck equations have found applications in areas such as plasma physics [1–3], surface physics [4–6], population dynamics [7], biophysics [8–13], engineering [14,15], neurosciences [16–24], nonlinear hydrodynamics [25–29], polymer physics [30–32], laser physics [33], pattern formation [34–36], psychology [37] and marketing [38]. They have been used to describe systems that exhibit power-law distributions, cut-off distributions, quantum distributions, or anomalous diffusion [39–65]. In particular, it has been suggested to regard them as evolution equations for probability densities of systems characterized by nonextensive [66–68] and general entropies, e.g., [39,55,61,65,69]. In spite of the increasing interest in this topic, a theory of nonlinear Fokker-Planck equations has not yet been established. Likewise, a definition of the term "nonlinear Fokker-Planck equation" has not been given so far.

In the present manuscript, we will discuss general aspects of nonlinear Fokker-Planck equations. We will focus on two questions. What is the physics of systems that can be described by means of nonlinear Fokker-Planck equations and what is the mathematics of stochastic processes that can be described by means of nonlinear Fokker-Planck equations? In order to elaborate these questions, we will consider families of stochastic processes. Let $\mathbf{X}(t)$ denote a M-dimensional time-dependent random variable defined on the phase space Ω and the interval $t \in [t_0, \infty)$. Let us assume that $\mathbf{X}(t)$ is distributed like $u(\mathbf{x})$ at an initial time t_0. Then, we denote the probability density of $\mathbf{X}$ by

$$P(\mathbf{x}, t; u) = \langle \delta(\mathbf{x} - \mathbf{X}(t)) \rangle \tag{1}$$

with $P(\mathbf{x}, t_0; u(\mathbf{x})) = u(\mathbf{x})$, where $\langle \cdots \rangle$ denotes an ensemble averaging. Likewise, joint probability densities can be defined such as $P(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1; u(\mathbf{x})) = \langle \delta(\mathbf{x}_1 - \mathbf{X}(t_1)) \delta(\mathbf{x}_2 - \mathbf{X}(t_2)) \rangle$. A family of stochastic processes can then be described by means of the hierarchy of joint distributions

$$
\begin{aligned}
&P(\mathbf{x}, t; u(\mathbf{x})) \ , \\
&P(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1; u(\mathbf{x})) \ , \\
&\ldots \\
&P(\mathbf{x}_n, t_n; \ldots ; \mathbf{x}_1, t_1; u(\mathbf{x})) \ , \\
&\ldots
\end{aligned}
\tag{2}
$$

for $t \geq t_0$ and $t_n \geq t_{n-1} \geq \ldots \geq t_1 \geq t_0$. We can now define an evolution equation for the single time-point probability density $P(\mathbf{x}, t; u)$ by

$$
\begin{aligned}
\frac{\partial}{\partial t} P(\mathbf{x}, t; u) = & \underbrace{- \sum_{i=1}^{M} \frac{\partial}{\partial x_i} D_i(\mathbf{x}, t, P) P(\mathbf{x}, t; u)}_{Y_1} \\
& \underbrace{+ \sum_{i,k=1}^{M} \frac{\partial^2}{\partial x_i \partial x_k} D_{ik}(\mathbf{x}, t, P) P(\mathbf{x}, t; u)}_{Y_2}
\end{aligned}
\tag{3}
$$

with $t \geq t_0$. We refer to Eq. (3) as a nonlinear Fokker-Planck equation because for $D_i(\mathbf{x}, t, P) = D_i(\mathbf{x}, t)$ and $D_{ik}(\mathbf{x}, t, P) = D_{ik}(\mathbf{x}, t)$ it reduces to the form of the conventional linear Fokker-Planck equation [70,71]. In particular, we refer to D_i and D_{ik} as drift and diffusion coefficients, respectively. Likewise, we call Y_1 the drift term and Y_2 the diffusion term. Note that we do not claim that Eq. (3) represents the most general form of a nonlinear Fokker-Planck equation. However, a large variety of nonlinear Fokker-Planck equations that have been discussed in the literature can be cast into the form (3). In the univariate case, Eq. (3) reads

$$
\frac{\partial}{\partial t} P(x, t; u) = - \frac{\partial}{\partial x} D_1(x, t, P) P(x, t; u) + \frac{\partial^2}{\partial x^2} D_2(x, t, P) P(x, t; u) \tag{4}
$$

with $t \geq t_0$. Note that in the following sections we will address constraints that may be imposed on the coefficients $D_1(x, t, P)$, $D_2(x, t, P)$, $D_i(\mathbf{x}, t, P)$, and $D_{ik}(\mathbf{x}, t, P)$. While Sec. 2 is concerned with the physics of systems described by nonlinear Fokker-Planck equations, Sec. 3 is about the mathematics of processes described by nonlinear Fokker-Planck equations. In particular, in Sec. 3 we will distinguish between strong and weak nonlinear Fokker-Planck equations and linear and nonlinear families of Markov processes.

2 On the physics of nonlinear Fokker-Planck equations

2.1 Stochastic feedback, hitchhiker processes, and path integral solutions

2.1.1 Stochastic feedback

We refer to a system described by a state variable $\mathbf{X}(t)$ that satisfies the evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t) = \mathbf{D}(\mathbf{X},t) + G(\mathbf{X},t)\cdot\mathbf{\Gamma}(t) \tag{5}$$

as a system with a feedback because the evolution of $\mathbf{X}$ depends on the state variable $\mathbf{X}$. Here, $\mathbf{D}$ is given by $\mathbf{D} = (D_1, \ldots, D_M)$, G describes a $M \times M$ matrix of noise amplitudes, and $\mathbf{\Gamma}$ describes a fluctuation force. For appropriately defined G and $\mathbf{\Gamma}$, the stochastic evolution of $\mathbf{X}$ can be described in terms of the linear Fokker-Planck equation [71]

$$\frac{\partial}{\partial t}P(\mathbf{x},t;u) = -\sum_{i=1}^{M}\frac{\partial}{\partial x_i}D_i(\mathbf{x},t)P(\mathbf{x},t;u) + \sum_{i,k=1}^{M}\frac{\partial^2}{\partial x_i \partial x_k}D_{ik}(\mathbf{x},t)P(\mathbf{x},t;u)\ . \tag{6}$$

In this picture, the dependency of the coefficients $D_i(\mathbf{x},t)$ and $D_{ik}(\mathbf{x},t)$ on the state vector $\mathbf{x}$ reflects the fact that we deal with a feedback system for which the actual state of the system affects the dynamics of the system. Now, let us write Eq. (3) as

$$\begin{aligned}\frac{\partial}{\partial t}P(\mathbf{x},t;u) = &-\sum_{i=1}^{M}\frac{\partial}{\partial x_i}D_i\left(\mathbf{x},t;z\right)\Big|_{z\,=\,P}P(\mathbf{x},t;u)\\ &+\sum_{i,k=1}^{M}\frac{\partial^2}{\partial x_i \partial x_k}D_{ik}(\mathbf{x},t;z)\Big|_{z\,=\,P}P(\mathbf{x},t;u)\ .\end{aligned} \tag{7}$$

Here, the drift and diffusion coefficients depend on an ensemble measure, namely, the probability density P. Consequently, not only depends the evolution of the system described by Eq. (7) on the state of the system but also on stochastic properties of the system. Both the state variable and the probability density are fed back into the evolution equation that determines the stochastic behavior of the system. In analogy to the deterministic feedback loop involved in Eq. (5), we may say that systems described by Eq. (7) exhibit a stochastic (or statistical) feedback loop [42,57,72,73]. Next, we will show that stochastic feedback processes may be interpreted in terms of hitchhiker processes and path integral solutions.

2.1.2 Hitchhiker processes

Let us consider Eq. (3) for coefficients D_i and D_{ik} that do not explicitly depend on time:

$$\frac{\partial}{\partial t}P(\mathbf{x},t;u) = \\ -\sum_{i=1}^{M}\frac{\partial}{\partial x_i}D_i\left(\mathbf{x};P(\mathbf{x},t;u)\right)P(\mathbf{x},t;u) + \sum_{i,k=1}^{M}\frac{\partial^2}{\partial x_i \partial x_k}D_{ik}(\mathbf{x};P(\mathbf{x},t;u))P(\mathbf{x},t;u)\ . \tag{8}$$

By means of the coefficients $D_i(\mathbf{x},\cdot)$ and $D_{ik}(\mathbf{x},\cdot)$ we can also define a linear Fokker-Planck equation for the probability density W:

$$\frac{\partial}{\partial t}W(\mathbf{x},t;u)_{t_0} = \\ -\sum_{i=1}^{M}\frac{\partial}{\partial x_i}D_i(\mathbf{x};u)W(\mathbf{x},t;u)_{t_0} + \sum_{i,k=1}^{M}\frac{\partial^2}{\partial x_i \partial x_k}D_{ik}(\mathbf{x};u)W(\mathbf{x},t;u)_{t_0}\ . \tag{9}$$

Eq. (9) defines for every initial distribution u a time-dependent probability density $W(\mathbf{x},t;u)_{t_0}$ satisfying $W(\mathbf{x},z;u)_z = u$. Note that we require now that Eq. (9) indeed corresponds to a linear Fokker-Planck equations, which implies, for example, that D_{ik} can be interpreted as the second Kramers-Moyal coefficient of a stochastic process and, consequently, describes a symmetric and semi-positive definite matrix [71]. In Fig. 1 we have plotted schematically three solution $W^* = W(\mathbf{x}, t; u^*)_{t_0}$, $W^1 = W(\mathbf{x}, t; u_1)_{t_1}$, and $W^2(\mathbf{x}, t; u_2)_{t_2}$ of Eq. (9) for the initial distributions u^*, u_1, and u_2.

Insert Figure 1 about here

Comparing Eqs. (8) and (9) we realize that the equivalence

$$\frac{\partial}{\partial t} P(\mathbf{x}, t; u) = \frac{\partial}{\partial t} W(\mathbf{x}, t; P)_t \tag{10}$$

holds. Integrating Eq. (10) with respect to t, gives us

$$P(\mathbf{x}, t; u) = u(\mathbf{x}) + \int_{t_0}^{t} \frac{\partial}{\partial s} W(\mathbf{x}, s; P(\mathbf{x}, s; u))_s \, \mathrm{d}s \ . \tag{11}$$

Eqs. (8,...,11) can be interpreted as follows. On the one hand, we deal with Markov processes B^i described by the transient probability densities $W(\mathbf{x}, t; u_i)_{t_i}$ for different initial distributions u_i and initial times t_i. On the other hand, we have a process A described by the nonlinear Fokker-Planck equation (8) for a particular initial distribution, say, u^*. Then, let us observe A stroboscopically at times $t_i = t_0 + \Delta t$. We find that at every instance t_i the process A evolves like the Markov process $W^i = W(\mathbf{x}, t, u_i)_{t_i}$ with the initial distribution $u_i = P(\mathbf{x}, t_i, u^*)$. However, at every instance t_i we deal with a different initial distribution u_i and, consequently, with a different Markov process W^i. In other words, the process A jumps from the Markov process B^i with probability density $W^i = W(\mathbf{x}, t, u_i)_{t_i}$ and $u_i = P(\mathbf{x}, t_i, u^*)$ at time t_i to the Markov process B^{i+1} with distribution $W^{i+1} = W(\mathbf{x}, t; u_{i+1})_{t_{i+1}}$ and initial distribution $u_{i+1} = P(\mathbf{x}, t_{i+1}, u^*)$ at time t_{i+1}. The stochastic process A described by the nonlinear Fokker-Planck equation (8) evolves within the family of Markov processes with probability densities $W(\mathbf{x}, t; u_i)_{t_i}$ defined by the linear Fokker-Planck equation (9). We may also say that the process A "rides on the back" of the Markov processes B^i. Since A permanently changes its host processes

B^i, the behavior of A resembles the one of a hitchhiker, which is the reason why one may refer to processes with stochastic feedback as hitchhiker processes. From Fig. 1 we can also read off that the change $\Delta P(\mathbf{x}, t; u)$ of P can approximately compute from $\Delta W(\mathbf{x}, t; P)_t$ for a small interval Δt. This approximation becomes exact in the limit of $\Delta t \to 0$, which is expressed by Eqs. (10) and (11).

2.1.3 Path integral solutions

Hitchhiker processes can also be described in terms of path integral solutions of nonlinear Fokker-Planck equations. Let us illustrate this issue for a univariate nonlinear Fokker-Planck equation. For $t \geq t_0$ and small time steps τ the evolution of P can approximately be determined by means of the integral equation

$$P(x, t+\tau; u) \approx \int_{\Omega} G(x, x'; t, \tau, P) P(x', t; u) \, \mathrm{d}x' \ , \tag{12}$$

where $G(x, x'; t, \tau; P)$ is called short time propagator and is defined by [74]

$$\begin{aligned} & G(x, x'; t, \tau; P) \\ & = \sqrt{\frac{1}{2\pi\tau D_2[x', t; P(x', t; u)]}} \exp\left\{ -\frac{[x - x' - \tau D_1[x', t; P(x', t; u)]]^2}{2\tau D_2[x', t; P(x', t; u)]} \right\} \ . \end{aligned} \tag{13}$$

Taking infinitesimal small time steps τ into account, we obtain

$$P(x, t; u) = \lim_{N \to \infty, \tau \to 0} \int_{\Omega^N} P(x_1, t_1; u) \prod_{i=1}^{N} \left\{ G(x_{i+1}, x_i; t_i, \tau, P) \, \mathrm{d}x_i \right\} \tag{14}$$

with $\tau N = t - t_1$, where x corresponds to the final chain element x_{N+1} in the limit $N \to \infty$. Eq. (14) is the path integral solution of the nonlinear Fokker-Planck equation (4) [74], which can in particular be used for numerical computations [74,75]. Eq. (14) generalizes the path integral approach for the linear Fokker-Planck equation which can be found, for example, in [70,76–78]. The path integral solution (14) tells us that the evolution of a stochastic process described by the nonlinear Fokker-Planck equation (4) is governed at time t_1 and for a small time step by a short time propagator $G_1 = G(x, x'; t_1, \tau, P)$ that belongs to the distribution P of that process at the time t_1. At an ensuing moment t_2 the evolution is governed by another short time propagator $G_2 = G(x, x'; t_2, \tau, P)$ belonging to the process distribution P at time t_2. That is, the process permanently switches between different short time propagator which is in line with the notions of hitchhiker processes. Moreover, the system is characterized by a probability-dependent short-time propagator which reflects the fact that it exhibits a stochastic feedback structure.

3 On the mathematics of nonlinear Fokker-Planck equations

3.1 Linear versus nonlinear families of Markov processes

Although the nonlinear Fokker-Planck equation (3) defines the evolution of the probability density $P(\mathbf{x}, t; u)$ it cannot be used to define the hierarchy of probability densities shown in Eq. (2). That is, from Eq. (3) we cannot compute a stochastic process [46]. Taking a mathematical point of view, our objective now is to derive stochastic processes that are consistent with the nonlinear Fokker-Planck equation (3). To this end, it is useful to distinguish between linear and nonlinear families of Markov processes.

To begin with, let us define conditional probability densities for families of stochastic processes by

$$P(\mathbf{x}_n, t_n | \mathbf{x}_{n-1}, t_{n-1}; \ldots; \mathbf{x}_1, t_1; u) = \frac{P(\mathbf{x}_n, t_n; \ldots; \mathbf{x}_1, t_1; u)}{P(\mathbf{x}_{n-1}, t_{n-1}; \ldots; \mathbf{x}_1, t_1; u)} \ . \quad (15)$$

If the conditional probability density (15) satisfies

$$P(\mathbf{x}_n, t_n|\mathbf{x}_{n-1}, t_{n-1}; \ldots; \mathbf{x}_1, t_1; u) = P(\mathbf{x}_n, t_n|\mathbf{x}_{n-1}, t_{n-1}; u) \tag{16}$$

for all n, then we deal with a family of Markov processes. It is important to realize that every member of this family of conditional probability densities depends only on two time points. For example, let us consider a family of stochastic processes characterized by the initial distributions u_1, u_2, u_3 and so on. Then, we deal with the set of Markov transition probability densities given by $P^i(\mathbf{x}_n, t_n|\mathbf{x}_{n-1}, t_{n-1}; \ldots; \mathbf{x}_1, t_1) = P^i(\mathbf{x}_n, t_n|\mathbf{x}_{n-1}, t_{n-1})$. Next, we distinguish between linear and nonlinear families of Markov processes. For linear families there is a unique transition probability density

$$P(\mathbf{x}, t|\mathbf{x}', t'; u) = P(\mathbf{x}, t|\mathbf{x}', t') \tag{17}$$

that describes the evolution of all members of the family. In other words, if the transition probability density of a family of Markov processes does not depend on the initial distribution u, we deal with a linear family. If Eq. (17) is not satisfied, we referred to the family as nonlinear, see also Table 1.

Insert Table 1 about here

3.1.1 Linear families of Markov processes

For linear families of Markov processes the integral equation

$$P(\mathbf{x}, t; u) = \int P(\mathbf{x}, t|\mathbf{x}', t') P(\mathbf{x}', t'; u) \, \mathrm{d}^M x' \tag{18}$$

and, in particular, the propagator relation

$$P(\mathbf{x}, t; u) = \int P(\mathbf{x}, t|\mathbf{x}', t_0) \; u(\mathbf{x}') \, \mathrm{d}^M x' \tag{19}$$

are satisfied for $t \geq t' \geq t_0$. Let us assume that the integral equation (18) can be transformed into a partial differential equation of the form

$$\frac{\partial}{\partial t} P(\mathbf{x}, t; u) = A[P] \ , \qquad (20)$$

where A denotes a differential operator that involves derivatives with respect to $\mathbf{x}$. In line with a pervious study [79], we can then show that the operator A is linear with respect to P, which implies that the evolution equation (20) is linear with respect to P. That is, linear families of Markov processes are related to linear partial differential equations of the form (20). Consequently, processes described by nonlinear Fokker-Planck equations such as Eq. (3) cannot be interpreted in terms of linear families of Markov processes. Furthermore, for linear families of Markov processes we obtain Fokker-Planck equations of the form [71]

$$\begin{aligned} &\frac{\partial}{\partial t} P(\mathbf{x}, t; u) = \\ &-\sum_{i=1}^{M} \frac{\partial}{\partial x_i} D_i(\mathbf{x}, t) P(\mathbf{x}, t; u) + \sum_{i,k=1}^{M} \frac{\partial^2}{\partial x_i \partial x_k} D_{ik}(\mathbf{x}, t) P(\mathbf{x}, t; u) \ , \\ &\frac{\partial}{\partial t} P(\mathbf{x}, t | \mathbf{x}', t') = \\ &-\sum_{i=1}^{M} \frac{\partial}{\partial x_i} D_i(\mathbf{x}, t) P(\mathbf{x}, t | \mathbf{x}', t') + \sum_{i,k=1}^{M} \frac{\partial^2}{\partial x_i \partial x_k} D_{ik}(\mathbf{x}, t) P(\mathbf{x}, t | \mathbf{x}', t') \end{aligned} \qquad (21)$$

and solutions $P(\mathbf{x}, t; u)$ that satisfy the superposition principle

$$pP(\mathbf{x}, t, u_1) + (1-p)P(\mathbf{x}, t, u_2) = P(\mathbf{x}, t, pu_1 + (1-p)u_2) \qquad (22)$$

with $p \in [0, 1]$.

3.1.2 Nonlinear families of Markov processes

For nonlinear families of Markov processes, from Eq. (16) it follows that the Chapman-Kolmogorov equation reads

$$P(\mathbf{x}_3, t_3|\mathbf{x}_1, t_1; u) = \int_\Omega P(\mathbf{x}_3, t_3|\mathbf{x}_2, t_2; u) P(\mathbf{x}_2, t_2|\mathbf{x}_1, t_1; u)\, \mathrm{d}^M x_2 \ . \quad (23)$$

Consequently, the Kramers-Moyal expansion [71] gives us

$$\frac{\partial}{\partial t} P(\mathbf{x}, t|\mathbf{x}', t'; u) = \sum_{n=1}^{\infty} (-1)^n \sum_{i_1,\ldots,i_n=1}^{M} \frac{\partial^n}{\partial x_{i_1} \cdots \partial x_{i_n}} \left[D_{i_1,\ldots,i_n}(\mathbf{x}, t; u) P(\mathbf{x}, t|\mathbf{x}', t'; u) \right] \quad (24)$$

with the Kramers-Moyal coefficients defined by

$$\begin{aligned} M_{i_1,\ldots,i_n}(\mathbf{x}', t, t'; u) &= \int_\Omega (y_{i_1} - x_{i_1}) \cdots (y_{i_n} - x_{i_n})\, P(\mathbf{y}, t|\mathbf{x}', t'; u)\, \mathrm{d}^M y \ , \\ D_{i_1,\ldots,i_n}(\mathbf{x}, t; u) &= \frac{1}{n!} \lim_{\tau \to 0} \frac{M_{i_1,\ldots,i_n}(\mathbf{x}, t+\tau, t; u)}{\tau} \ , \\ &= \frac{1}{n!} \lim_{\tau \to 0} \int_\Omega (y_{i_1} - x_{i_1}) \cdots (y_{i_n} - x_{i_n})\, P(\mathbf{y}, t+\tau|\mathbf{x}, t; u)\, \mathrm{d}^M y \ . \end{aligned} \quad (25)$$

In contrast to a linear family of Markov processes, the Kramers-Moyal coefficients now depend on u. The Pawula theorem [71,80] applies to any transition probability density and, consequently, to the transition probability density (16) as well. Therefore, nonlinear families of Markov processes with a finite number of nonvanishing Kramers-Moyal coefficients are described by

$$\begin{aligned} \frac{\partial}{\partial t} P(\mathbf{x}, t; u) &= - \sum_{i=1}^{M} \frac{\partial}{\partial x_i} D_i(\mathbf{x}, t; u) P(\mathbf{x}, t; u) + \sum_{i,k=1}^{M} \frac{\partial^2}{\partial x_i \partial x_k} D_{ik}(\mathbf{x}, t; u) P(\mathbf{x}, t; u) \ , \\ \frac{\partial}{\partial t} P(\mathbf{x}, t|\mathbf{x}', t'; u) &= - \sum_{i=1}^{M} \frac{\partial}{\partial x_i} D_i(\mathbf{x}, t; u) P(\mathbf{x}, t|\mathbf{x}', t'; u) \\ &\quad + \sum_{i,k=1}^{M} \frac{\partial^2}{\partial x_i \partial x_k} D_{ik}(\mathbf{x}, t; u) P(\mathbf{x}, t|\mathbf{x}', t'; u) \ . \end{aligned} \quad (26)$$

The stochastic processes defined by Eq. (26) can then be described in terms of hierarchies of probability densities given by

$$P(\mathbf{x},t;u) = \int_\Omega P(\mathbf{x},t|\mathbf{x}',t_0;u)\,u(\mathbf{x}')\,\mathrm{d}^M x' \ ,$$
$$P(\mathbf{x}_n,t_n;\ldots;\mathbf{x}_1,t_1;u)$$
$$= P(\mathbf{x}_n,t_n|\mathbf{x}_{n-1},t_{n-1};u)\cdots P(\mathbf{x}_2,t_2|\mathbf{x}_1,t_1;u)P(\mathbf{x}_1,t_1;u) \ . \tag{27}$$

Note that the evolution equation (26) for $P(\mathbf{x},t,u)$ is nonlinear with respect to u. Consequently, the superposition principle (22) does not necessarily hold.

3.2 Markov embedding of nonlinear Fokker-Planck equations

We are now in the position to define stochastic processes on the basis of the nonlinear Fokker-Planck equation (3). To this end, we define for $t \geq t' \geq t_0$ a family of transition probability densities by

$$\frac{\partial}{\partial t}P(\mathbf{x},t|\mathbf{x}',t';u) = -\sum_{i=1}^{M}\frac{\partial}{\partial x_i}D_i(\mathbf{x},t,P(\mathbf{x},t;u))P(\mathbf{x},t|\mathbf{x}',t';u)$$
$$+\sum_{i,k=1}^{M}\frac{\partial^2}{\partial x_i\partial x_k}D_{ik}(\mathbf{x},t,P(\mathbf{x},t;u))P(\mathbf{x},t|\mathbf{x}',t';u) \ , \tag{28}$$

where $P(\mathbf{x},t;u)$ corresponds to the solution of Eq. (3). Note that Eqs. (3) and (28) have the same drift and diffusion coefficients. If for solutions $P(\mathbf{x},t;u)$ the coefficients $D_i(\mathbf{x},t,P(\mathbf{x},t;u))$ and $D_{ik}(\mathbf{x},t,P(\mathbf{x},t;u))$ describe the first and second Kramers-Moyal coefficients of a nonlinear family of Markov processes, then we will refer to Eq. (3) as a strong nonlinear Fokker-Planck equation. Otherwise, we will call Eq. (3) a weak nonlinear Fokker-Planck equation. For strong nonlinear Fokker-Planck equations a family of stochastic processes can be defined by means of Eqs. (3), (27) and (28).

Let us attack this issue from another perspective. Eqs. (3) and (28) can equivalently be expressed as

$$\frac{\partial}{\partial t}P(\mathbf{x},t;u) = -\sum_{i=1}^{M}\frac{\partial}{\partial x_i}D'_i(\mathbf{x},t,t_0,u))P(\mathbf{x},t;u) + \sum_{i,k=1}^{M}\frac{\partial^2}{\partial x_i \partial x_k}D'_{ik}(\mathbf{x},t,t_0,u)P(\mathbf{x},t;u) \tag{29}$$

and

$$\frac{\partial}{\partial t}P(\mathbf{x},t|\mathbf{x}',t';t_0) = -\sum_{i=1}^{M}\frac{\partial}{\partial x_i}D'_i(\mathbf{x},t,t_0,u))P(\mathbf{x},t|\mathbf{x}',t';u) + \sum_{i,k=1}^{M}\frac{\partial^2}{\partial x_i \partial x_k}D'_{ik}(\mathbf{x},t,t_0,u)P(\mathbf{x},t|\mathbf{x}',t';u) \tag{30}$$

with

$$D'_i(\mathbf{x},t,t_0,u) = D_i(\mathbf{x},t,P(\mathbf{x},t,u)) \ , \quad D'_{ik}(\mathbf{x},t,t_0,u) = D_{ik}(\mathbf{x},t,P(\mathbf{x},t,u)) \ . \tag{31}$$

If Eqs. (29,...,31) define for solutions of Eq. (3) a nonlinear family of Markov diffusion processes then Eq. (3) is referred to as a strong nonlinear Fokker-Planck equation. Otherwise, we consider Eq. (3) as a weak nonlinear Fokker-Planck equation. A weak nonlinear Fokker-Planck equations primarily defines the evolution of a probability density $P(\mathbf{x},t;u)$ of a stochastic process but does not provide us with information about other members of the hierarchy (2). In contrast, strong nonlinear Fokker-Planck equations can be used to construct well-defined stochastic processes for which all members of the hierarchy (2) can be determined.

A sufficient condition that Eqs. (29,...,31) define a nonlinear family of Markov diffusion processes is that Eq. (30) has a fundamental solution, that is, it has solutions satisfying $P(\mathbf{x},t|\mathbf{x}',t';u) \geq 0$, $\int_\Omega P(\mathbf{x},t|\mathbf{x}',t';u)\,\mathrm{d}^M x = 1$, and $\lim_{t\to t'} P(\mathbf{x},t|\mathbf{x}',t';u) = \delta(\mathbf{x}-\mathbf{x}')$, see, for example, [81, Sec. 2.3.5] or [82, Sec 5]. In other words, in order to prove that we deal with a strong nonlinear Fokker-Planck equation of the form (3) we need to show that for transient solutions of Eq. (3) the solutions of Eq. (28) (or Eq. (30)) indeed describe transition probability densities. A necessary condition that Eqs. (29,...,31) define a nonlinear family of Markov diffusion processes and that Eq. (3) represents a strong nonlinear Fokker-Planck equations is that the diffusion coefficient D_{ik} corresponds to a symmetric and semi-positive definite matrix (which are the conditions satisfied by second Kramers-Moyal coefficients).

3.3 Langevin equations for strong nonlinear Fokker-Planck equations

It is clear that if we can define Markov diffusion processes by means of the nonlinear Fokker-Planck equation (3) then we can also assign a Langevin equation to Eq. (3). We first need to define a matrix G with elements $G_{ik}(\mathbf{x}, t, P)$ that satisfies

$$\sum_{l=1}^{M} G_{il}(\mathbf{x}, t, P) G_{lk}(\mathbf{x}, t, P) = D_{ik}(\mathbf{x}, t, P) \tag{32}$$

(for an explicit derivation of G_{il} see, for example, [71]). Then, stochastic trajectories $\mathbf{X}(t)$ of $P(\mathbf{x}, t; u)$ defined by Eqs. (29,...,31) are described by the Ito-Langevin equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} X_i(t) = D_i'(\mathbf{X}, t, t_0, u) + \sum_{k=1}^{M} G_{ik}'(\mathbf{X}, t; u) \Gamma_k(t) \ , \\ D_i'(\mathbf{x}, t, t_0, u) = D_i(\mathbf{x}, t, P(\mathbf{x}, t; u)) \ , \\ G_{ik}'(\mathbf{x}, t, t_0, u) = G_{ik}(\mathbf{x}, t, P(\mathbf{x}, t; u)) \ , \end{aligned} \tag{33}$$

where P denotes a solution of Eq. (3). Just as in the univariate case, a closed description can be found due to the equivalence between the probability densities P obtained from the Ito-Langevin equation (33) and the nonlinear Fokker-Planck equation (3). In line with earlier suggestions [39,46], this closed description reads

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) = D_i'(\mathbf{X}, t, t_0, u) + \sum_{k=1}^{M} G_{ik}'(\mathbf{X}, t, t_0, u) \Gamma_k(t) \ ,$$

$$D_i'(\mathbf{x}, t, t_0, u) = D_i(\mathbf{x}, t, \langle \delta(\mathbf{x} - \mathbf{X}(t)) \rangle) \ ,$$
$$G_{ik}'(\mathbf{x}, t, t_0, u) = G_{ik}(\mathbf{x}, t, \langle \delta(\mathbf{x} - \mathbf{X}(t)) \rangle) \ . \tag{34}$$

3.4 Transition probability densities versus transient probability densities

One of the key issues that advances our understanding of nonlinear Fokker-Planck equations is the relationship between transition probability densities and transient probability densities. Let us first elucidate this relationship for linear Fokker-Planck equations. To this end, we consider a Markov diffusion process described by

$$\begin{aligned}
&\frac{\partial}{\partial t} P(\mathbf{x}, t; u) = \\
&-\sum_{i=1}^{M} \frac{\partial}{\partial x_i} D_i(\mathbf{x}, t, t_0) P(\mathbf{x}, t; u) + \sum_{i,k=1}^{M} \frac{\partial^2}{\partial x_i \partial x_k} D_{ik}(\mathbf{x}, t, t_0) P(\mathbf{x}, t; u) \ , \\
&\frac{\partial}{\partial t} P(\mathbf{x}, t | \mathbf{x}', t'; u) = -\sum_{i=1}^{M} \frac{\partial}{\partial x_i} D_i(\mathbf{x}, t, t_0) P(\mathbf{x}, t | \mathbf{x}', t'; u) \\
&\qquad + \sum_{i,k=1}^{M} \frac{\partial^2}{\partial x_i \partial x_k} D_{ik}(\mathbf{x}, t, t_0) P(\mathbf{x}, t | \mathbf{x}', t'; u)
\end{aligned} \tag{35}$$

for $t \geq t' \geq t_0$. Note that here the drift and diffusion coefficients depend explicitly on the parameter t_0 which corresponds to the initial time of the process. As a simple example we may think of a periodically driven overdamped motion of a particle given by $\mathrm{d}X(t)/\mathrm{d}t = -\gamma x + A \sin[\Omega(t - t_0)] + \sqrt{Q}\Gamma(t)$ for $t \geq t_0$. It is important to realize that the fact that D_i and D_{ik} depend on t_0 does not violate the requirement that we deal with a Markov process (see also the example). In a similar vain, we would like to emphasis that from Eq. (35) it follows that $P(\mathbf{x}, t; u)$ and $P(\mathbf{x}, t | \mathbf{x}', t'; u)$ depend on the parameter t_0 such that we could also write $P(\mathbf{x}, t; u) = P(\mathbf{x}, t; u, t_0)$ and $P(\mathbf{x}, t | \mathbf{x}', t'; u) = P(\mathbf{x}, t | \mathbf{x}', t'; u, t_0)$ or alternatively $P(\mathbf{x}, t; u) = P(\mathbf{x}, t; u)_{t_0}$ and $P(\mathbf{x}, t | \mathbf{x}', t'; u) = P(\mathbf{x}, t | \mathbf{x}', t'; u)_{t_0}$. We prefer the notation $P(\mathbf{x}, t | \mathbf{x}', t'; u)_{t_0}$ (and likewise $P(\mathbf{x}, t; u)_{t_0}$) because the term $P(\mathbf{x}, t | \mathbf{x}', t'; u, t_0)$ could wrongly be interpreted as a non-Markovian conditional probability density $P(\mathbf{x},t|\mathbf{x}',t';\mathbf{x}_{t_0},t_0)$. Just to avoid confusion: for every initial condition u the solution $P(\mathbf{x},t|\mathbf{x}',t';u)_{t_0}$ of Eq. (35) describes a Markov transition probability density. However, $P(\mathbf{x},t|\mathbf{x}',t';u)_{t_0}$ depends in general on all parameters that occur in D_i and D_{ik} and, consequently, on the parameter t_0.

By comparing the evolution equations for $P(\mathbf{x},t;u)$ and $P(\mathbf{x},t|\mathbf{x}',t';u)$ of Eq. (35), we can verify that the equivalence

$$P(\mathbf{x},t;\delta(\mathbf{x}-\mathbf{x}_0))_{t_0} = P(\mathbf{x},t|\mathbf{x}_0,t_0;\delta(\mathbf{x}-\mathbf{x}_0)) \tag{36}$$

holds. In words, the transient probability density $P(\mathbf{x},t;\delta(\mathbf{x}-\mathbf{x}_0))$ with initial distribution $\delta(\mathbf{x}-\mathbf{x}_0)$ can be computed from the transition probability density $P(\mathbf{x},t|\mathbf{x}_0,t_0;\delta(\mathbf{x}-\mathbf{x}_0))$. However, if D_i or D_{ik} depend explicitly on t_0 then for $t \geq t' > t_0$ we have

$$P(\mathbf{x},t|\mathbf{x}',t';\delta(\mathbf{x}-\mathbf{x}')) \neq P(\mathbf{x},t;\delta(\mathbf{x}-\mathbf{x}_0))_{t_0}|_{\mathbf{x}_0=\mathbf{x}',t_0=t'} \ . \tag{37}$$

This relation tells us that if we determine the transient probability density $P(\mathbf{x},t;\delta(\mathbf{x}-\mathbf{x}_0))_{t_0}$ of the stochastic process given by Eq. (35) and then replace $\mathbf{x}_0$ and t_0 by $\mathbf{x}'$ and t' then we do not obtain the transition probability density $P(\mathbf{x},t|\mathbf{x}',t';\delta(\mathbf{x}-\mathbf{x}'))$. That is, for the linear Fokker-Planck equation (35) with coefficients that explicitly depend on t_0 the transition probability density $P(\mathbf{x},t|\mathbf{x}',t';u)$ does not correspond to the transient probability density $P(\mathbf{x},t;u)$ for $u=\delta(\mathbf{x}-\mathbf{x}_0)$ and $t' > t_0$. In a nutshell, transient solutions with delta-distributed initial distributions are not necessarily equivalent to transition probability densities.

This difference between transient solutions and transition probability densities becomes crucial for strong nonlinear Fokker-Planck equation because strong nonlinear Fokker-Planck equations can be mapped to linear Fokker-Planck equations with drift and diffusion coefficients that depend on $P(\mathbf{x},t;u)$. Since

$P(\mathbf{x}, t; u)$ depends on the initial time t_0, strong nonlinear Fokker-Planck equations can be mapped to linear Fokker-Planck equations with drift and diffusion coefficients that depend on t_0. For this reason, transient solutions of nonlinear Fokker-Planck equations cannot be considered as transition probability densities of Markov diffusion processes and do not necessarily satisfy the Chapman-Kolmogorov equation. This can conveniently be illustrated for transient solutions that can be written in the form a Gaussian distribution

$$P(x, t; \delta(x - x_0)) = \frac{1}{\sqrt{2\pi K(t, t_0)}} \exp\left\{ -\frac{[x - x_0 m(t, t_0)]^2}{2K(t, t_0)} \right\} \tag{38}$$

with $\lim_{t \to t_0} K(t, t_0) = 0$. For example, in a pervious study [79] for two nonlinear Fokker-Planck equations explicit time-dependent solutions satisfying Eq. (38) have been derived and it has been shown that they violate the Chapman-Kolmogorov equation. We would like to point out that in [79] from this finding the conclusion has been drawn that nonlinear Fokker-Planck equations cannot describe Markov processes. As illustrated in Sec. 3.2 this conclusion needs to be revised: nonlinear Fokker-Planck equations cannot describe linear families of Markov processes but they can describe nonlinear families of Markov processes. In sum, transient solutions of strong nonlinear Fokker-Planck equations that describe Markov processes can violate the Chapman-Kolmogorov equation because they do not correspond to the transition probability densities of the Markov processes. Let us illustrate this point by an example.

3.5 Example: Shimizu-Yamada model

In line with work by *Shimizu* and *Yamada* [83,84] and others [85], we consider the mean field Fokker-Planck equation

$$\frac{\partial}{\partial t} P(x, t; u) = \frac{\partial}{\partial x} \left[\gamma x + \kappa (x - \langle X \rangle) \right] P + Q \frac{\partial^2}{\partial x^2} P \tag{39}$$

for $\kappa \geq 0$, $\gamma > 0$, and $t \geq t_0$. For $\kappa = 0$ Eq. (39) reduces to the linear Fokker-Planck equation of an Ornstein-Uhlenbeck process. In what follows, we will consider the case $\kappa > 0$. For the mean field model (39) an exact time-dependent solution can be found [79,85]:

$$P(x,t;\delta(x-x_0)) = \frac{1}{\sqrt{2\pi K(t)}} \exp\left\{-\frac{[x-x_0 m(t)]^2}{2K(t)}\right\} \tag{40}$$

with

$$m(t) = \exp\{-\gamma(t-t_0)\} \ , \tag{41}$$
$$K(t) = \frac{Q}{\gamma+\kappa}[1-e^{-2(\gamma+\kappa)(t-t_0)}] \ . \tag{42}$$

For $\gamma > 0$ the time-dependent Gaussian distribution converges to the stationary one in the long time limit. As shown in Sec. 3.2 the stochastic processes described by the mean field Fokker-Planck equation (39) can equivalently be expressed in terms of a nonlinear family of Markov diffusion processes with transition probability densities given by

$$\frac{\partial}{\partial t}P(x,t|x',t';u) = \frac{\partial}{\partial x}\left[\gamma x + \kappa(x - \langle X\rangle_{P(x,t;u)})\right] P(x,t|x',t';u) + Q\frac{\partial^2}{\partial x^2}P(x,t|x',t';u) \ . \tag{43}$$

From Eq. (43) it is clear that Eq. (39) corresponds to a strong nonlinear Fokker-Planck equation because the effective drift and diffusion terms are continuous functions with respect to x and t. In particular D_1' reads $D_1'(x,t,t_0) = -\gamma x - \kappa(x - M_1(t))$ with $M_1(t)$ satisfying $\mathrm{d}M_1(t)/\mathrm{d}t = -\gamma M_1(t)$. For the time-dependent probability density (40) we obtain $M_1(t) = x_0 \exp\{-\gamma(t-t_0)\}$ which leads to

$$\frac{\partial}{\partial t}P(x,t|x',t';\delta(x-x_0)) = \frac{\partial}{\partial x}\left[(\gamma+\kappa)x - \kappa x_0 e^{-\gamma(t-t_0)}\right] P + Q\frac{\partial^2}{\partial x^2}P \ . \tag{44}$$

Eq. (44) is a linear Fokker-Planck equation describing an Ornstein-Uhlenbeck process subjected to a time-dependent driving force $f(t-t_0) = \kappa x_0 \exp\{-\gamma(t-t_0)\}$. By means of a moving frame transformation, one can show that Eq. (44) is solved by

$$P(x,t|x',t';\delta(x-x_0)) = \sqrt{\frac{1}{2\pi K(t,t')}} \exp\left\{-\frac{[x-g(t,t',t_0)-x'm(t,t')]^2}{2K(t,t')}\right\} \tag{45}$$

with

$$m(t,t') = \exp\{-(\gamma+\kappa)(t-t')\} \ , \tag{46}$$

$$K(t,t') = \frac{Q}{\gamma+\kappa}[1-e^{-2(\gamma+\kappa)(t-t')}] \ , \tag{47}$$

$$g(t,t',t_0) = x_0 \left[\exp\{-\gamma(t-t_0)\} - \exp\{-(\gamma+\kappa)t + \gamma t_0 + \kappa t'\}\right] \ . \tag{48}$$

Comparing Eqs. (40,...,42) with (45,...,48), we realize that

$$P(x,t|x',t';\delta(x-x_0)) \neq P(x,t;\delta(x-x_0))_{t_0}|_{x_0=x',t_0=t'} \tag{49}$$

for $t' > t_0$. That is, for $t' > t_0$ transition and transient probability densities differ from each other, see Sec. 3.4. However, for $t' = t_0$ the expression $g(t,t',t_0) + x_0 m(t,t')$ becomes

$$\begin{aligned} & g(t,t_0,t_0) + x_0 m(t,t_0) \\ & = x_0 \left[\exp\{-\gamma(t-t_0)\} - \exp\{-(\gamma+\kappa)(t-t_0)\}\right] + x_0 \exp\{-(\gamma+\kappa)(t-t_0)\} \\ & = x_0 \exp\{-\gamma(t-t_0)\} = x_0 m(t) \end{aligned} \tag{50}$$

leading to

$$P(x,t|x_0,t_0;\delta(x-x_0)) = P(x,t;\delta(x-x_0))_{t_0} \ . \tag{51}$$

It has been shown that the transient solution given by Eqs. (40,...,42) violates the Chapman-Kolmogorov equation if it is interpreted as a transition probability density [79]. In contrast, the transition probability density given by Eqs. (45,…,48) satisfies the Chapman-Kolmogorov equation (which can be shown explicitly by a detailed calculation, see e.g. [86]). With Eqs. (40) and (45) at hand, we can explicitly describe Markov diffusion processes defined by Eqs. (39) and (43) for initial distributions $u(x_1) = \delta(x_1 - x_0)$:

$$\begin{aligned}&P(x_n, t_n; \ldots; x_1, t_1; u) = \\ &P(x_n, t_n | x_{n-1}, t_{n-1}; u) \cdots P(x_2, t_2 | x_1, t_1; u) P(x_1, t_1; u) \ .\end{aligned} \tag{52}$$

While Eq. (52) describes the stochastic processes under consideration in terms of distribution functions, the Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = -(\gamma + \kappa) X(t) + \kappa \langle X(t) \rangle + \sqrt{Q} \Gamma(t) \tag{53}$$

for $t \geq t_0$ and $X(t_0) = x_0$ describes the very same processes in terms of stochastic trajectories. In particular, from Eqs. (40) and (45) we can compute the probability density $P(x, t; x', t'; \delta(x - x_0))$ and the correlation function $\langle X(t) X(t') \rangle$. Thus, we obtain

$$\langle X(t) X(t') \rangle = g(t, t', t_0) M_1(t') + e^{-(\gamma+\kappa)(t-t')} [K(t') + M_1(t')^2] \tag{54}$$

with $M_1(t') = x_0 \exp\{-\gamma(t - t_0)\}$ and K and g, respectively, defined by Eqs. (42) and (48). We solve numerically Eq. (53) and compute $\langle X(t) X(t') \rangle$ by means of $\langle X(t) X(t') \rangle = L^{-1} \sum_{k=1}^{L} X^l(t) X^l(t')$ for large L, where $X^l(t)$ are realizations of $X(t)$, and compare the result with the analytical expression (54). Fig. 2 shows $M_1(t)$ and $\langle X(t) X(t') \rangle$ as predicted by the theory of nonlinear Fokker-Planck equations versus the corresponding quantities as obtained from a simulation of Eq. (53).

Insert Figure 2 about here

4 Conclusions

Common properties of nonlinear Fokker-Planck equations

We have discussed a generic nonlinear Fokker-Planck equation that includes as special cases a variety of nonlinear Fokker-Planck equations that have been examined in the literature. We have found that a system described by the nonlinear Fokker-Planck equation is characterized by a stochastic feedback: the stochastic behavior of the system depends on the stochastic properties of the system. Processes with stochastic feedback may be regarded as hitchhiker processes defined on families of Markov diffusion processes. These hitchhiker processes permanently switch from one member of the family to another. The choice of a family member depends on stochastic properties which reflects the feedback structure. In addition, following *Wehner* and *Wolfer* we have demonstrated that path integral solutions of nonlinear Fokker-Planck equations nicely illustrate the impact of stochastic feedback.

Taking a mathematical point of view, we have shown that under particular circumstances solutions of nonlinear Fokker-Planck equations can correspond to Markov processes. To this end, we have considered nonlinear families of Markov processes. We have shown that nonlinear Fokker-Planck equations can be used to define nonlinear families of Markov processes. In this case, we deal with strong nonlinear Fokker-Planck equations, Kramers-Moyal coefficients that depend on single time-point probability densities, and transition probability densities that depend on initial distributions. This interpretation of nonlinear Fokker-Planck equations seems to be consistent with the Kramers-Moyal expansion for nonlinear Fokker-Planck equations proposed by *Borland* [39]. Furthermore, we would like to point out that the hitchhiker processes discussed in Sec. 3.2 are related to a special case of our embedding procedure: for nonlinear Fokker-Planck equations with coefficients that do not explicitly depend on t we obtain Eqs. (9) and (10) from Eqs. (28,. . . .31).

Finally, we have demonstrated that in the context of nonlinear Fokker-Planck equations we carefully need to distinguish between transition probability densities and transient probability densities. This issue should be considered as a crucial one because the literature abounds with derivations of exact time-dependent solutions for nonlinear Fokker-Planck equations. Our point here is that without a careful discussion these solutions should exclusively be regarded as transient solutions and not be interpreted as transition probability densities.

How nonlinear can nonlinear Fokker-Planck equations be?

The nonlinear Fokker-Planck equations that have been considered in the previous sections are nonlinear with respect to single time-point probability densities. In order to assign Markov processes to these nonlinear Fokker-Planck equations, we have used nonlinear evolution equations for single time-point probability densities but linear evolution equations for transition probability densities. As we have argued in Sec. 3.2, we can then transform for every initial distribution a nonlinear Fokker-Planck equation into a linear Fokker-Planck equation. A different kind of nonlinear Fokker-Planck equation has been discussed by *Borland* [38]. *Borland* considered a nonlinear Fokker-Planck equation given by a nonlinear evolution equation for a transition probability density. Nonlinear Fokker-Planck equations of this kind have not been addressed in the present study. They may be considered as Fokker-Planck equations that are more nonlinear than the one considered in this manuscript. The study of nonlinear evolution equations as suggested by *Borland* can be regarded as a new challenge that will most probably lead us to stochastic processes different from those discussed here.

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Table 1

Linear and nonlinear families of Markov processes.

$P(\mathbf{x}_n, t_n \vert \mathbf{x}_{n-1}, t_{n-1}; u)$	type of family
$\forall u : P(\mathbf{x}_n, t_n \vert \mathbf{x}_{n-1}, t_{n-1})$	linear
otherwise	nonlinear

Figure captions:

Fig. 1: Hitchhiker behavior of a stochastic process defined by a nonlinear Fokker-Planck equation.

Fig. 2: $M_1(t)$ and $C(t,t') = \langle X(t)X(t')\rangle$ as functions of t. Solid lines represent analytical results obtained from Eqs. (41) and (54). Diamonds represent results obtained from a simulation of Eq (53). $\langle X(t)X(t')\rangle$ is given for $t \geq t'$. For $t < t'$ we have put $\langle X(t)X(t')\rangle = 0$. Parameters: $Q = 2$, $x_0 = 1$, $\gamma = \kappa = 1$, and $t' = 0.4$.

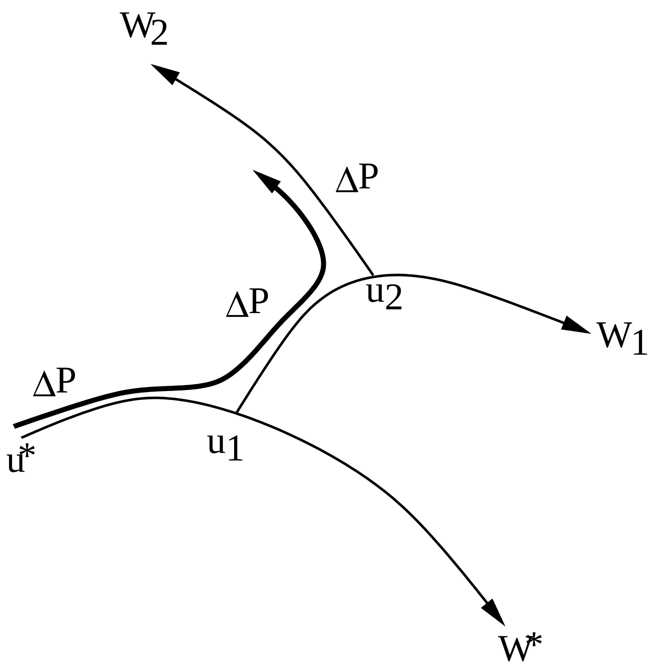
W2
ΔP
ΔP
u2
W1
ΔP
u*
u1
W*

Fig. 1.

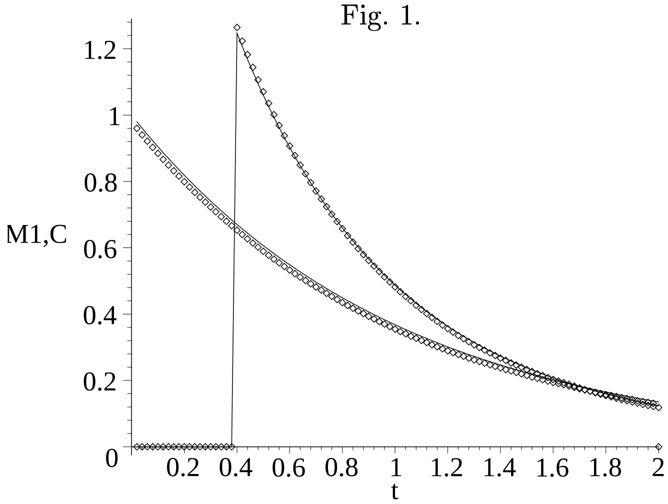
M1,C
1.2
1
0.8
0.6
0.4
0.2
0
0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2
t

Fig. 2.