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Workshop Ameland 2007

# Nonequilibrium phase transition in a system with chaotic dynamics

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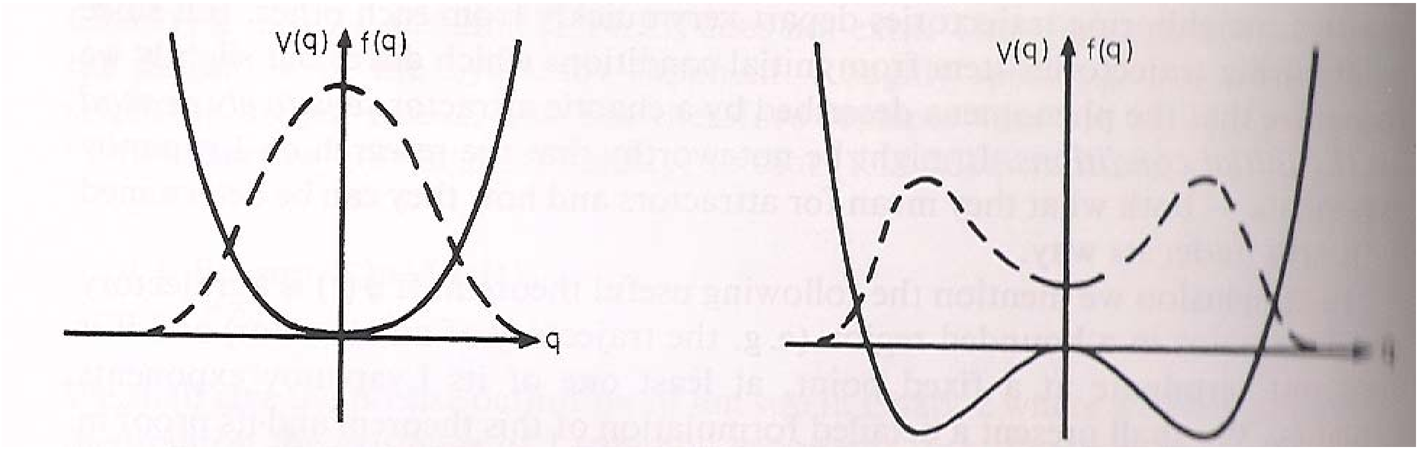
SS 2007

# Outline

1. Introduction
2. Observation of phase transition
3. Nature of instability (LE spectra, periodicity properties (Floquet))
4. Stochastic treatment of transition

# Nonequilibrium phase transition

- Fluctuation in systems far away from equilibrium

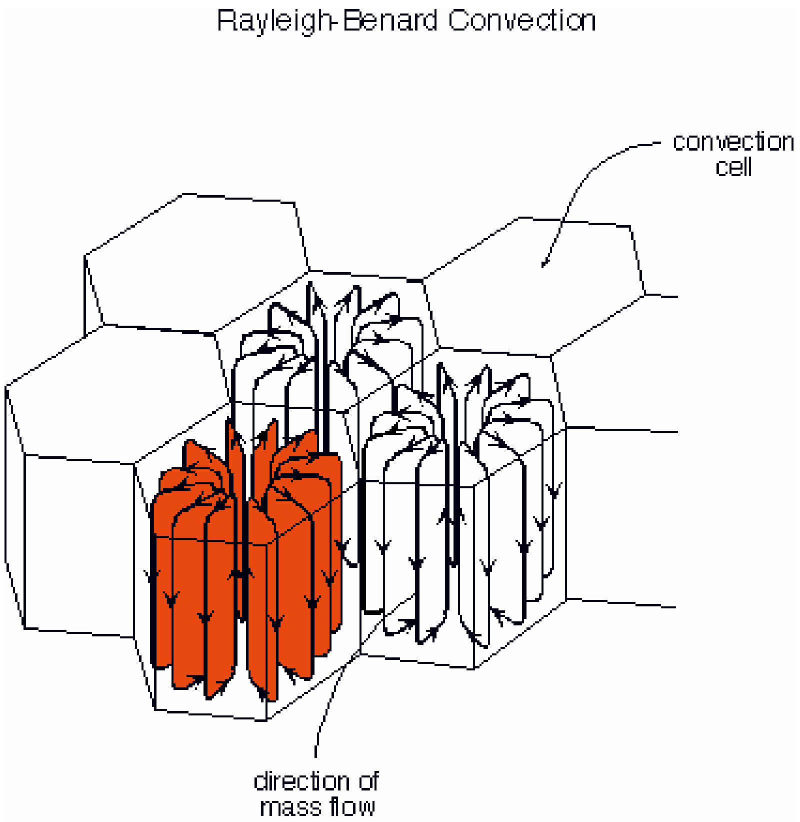



- Instability may lead to spatial, temporal or spatio-temporal behaviour
- Self-organisation as result of complex dynamics
- Profound understanding of instabilities of stationary, time periodic and quasiperiodic states (phys., biol., chem. systems)→ instabilities in chaotic systems

Motivation

# The ABCDE model

- Low-dimensional model for generation of dynamo effect (geo-dynamo, solar-dynamo, astrophysical objects)
- Investigation of electrically conducting fluid heated from below
- Convective motion (Rayleigh-Benard)
- Derivation from corres-ponding magnetohydro-dynamic equations by truncation of suitable mode expansion (analogue to Boussinesq→Lorenz)



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# Underlying equations
# ABCDE model

$$\dot{b}_1 = -\epsilon a_1 b_1 + \alpha x b_2$$

$$\dot{b}_2 = -\epsilon a_2 b_2 + \alpha x b_1$$

$$\dot{x} = \sigma(-x + y) - b_1 b_2$$

$$\dot{y} = -y + (r - z)x$$

$$\dot{z} = -bz + xy$$

- Lorenz equations nonlinearly coupled to two additional degrees of freedom related to magnetic field modes
- Excitation of additional degrees of freedom due to instability resulting in dynamo action
- Saturation of magnetic field as result of back coupling (might lead to non-magnetic convection)
- Near transition, intermittent behaviour of magnetic field modes

# Nonequilibrium phase transition in the ABCDE Model – numerical results

Estimation of onset of instability by means of characteristic exponent

Transformation into hyperbolic coordinates

$$b_1 = r\cosh(\phi), \quad b_2 = r\sinh(\phi)$$

Gives rise to

$$d_t r = \epsilon r[-a_1 + (a_2 - a_1)\sinh^2(\phi)]$$
$$d_t \phi = -\epsilon(a_2 - a_1)\sinh(\phi)\cosh(\phi) + \alpha x$$

Characteristic exponent (no back coupling on Lorenz equation):

$$r(t) = r(0)\exp[\Lambda(t)t]$$
$$\Lambda(t) = \epsilon(-a_1 + (a_2 - a_1)\frac{1}{t}\int_0^t \sinh^2\phi_0(\tau)d\tau)$$

# Nonequilibrium phase transition in the ABCDE Model – numerical results

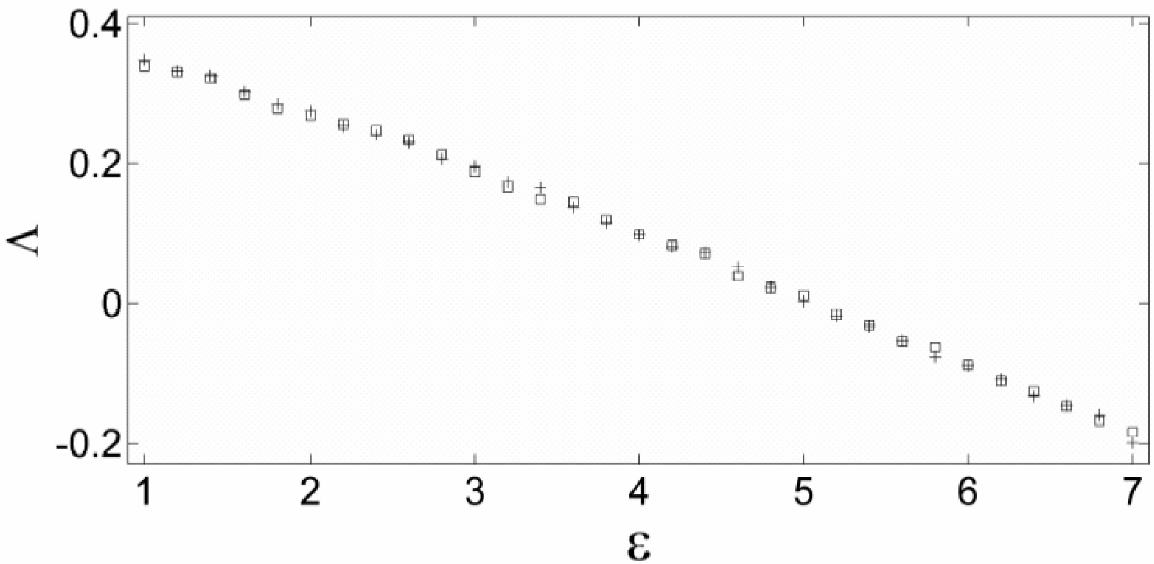



**Figure 4.1:** Characteristic exponent $\Lambda$ as function of the control parameter $\epsilon$ for two different initial conditions $(+, \square)$

# Nonequilibrium phase transition in the ABCDE Model – numerical results

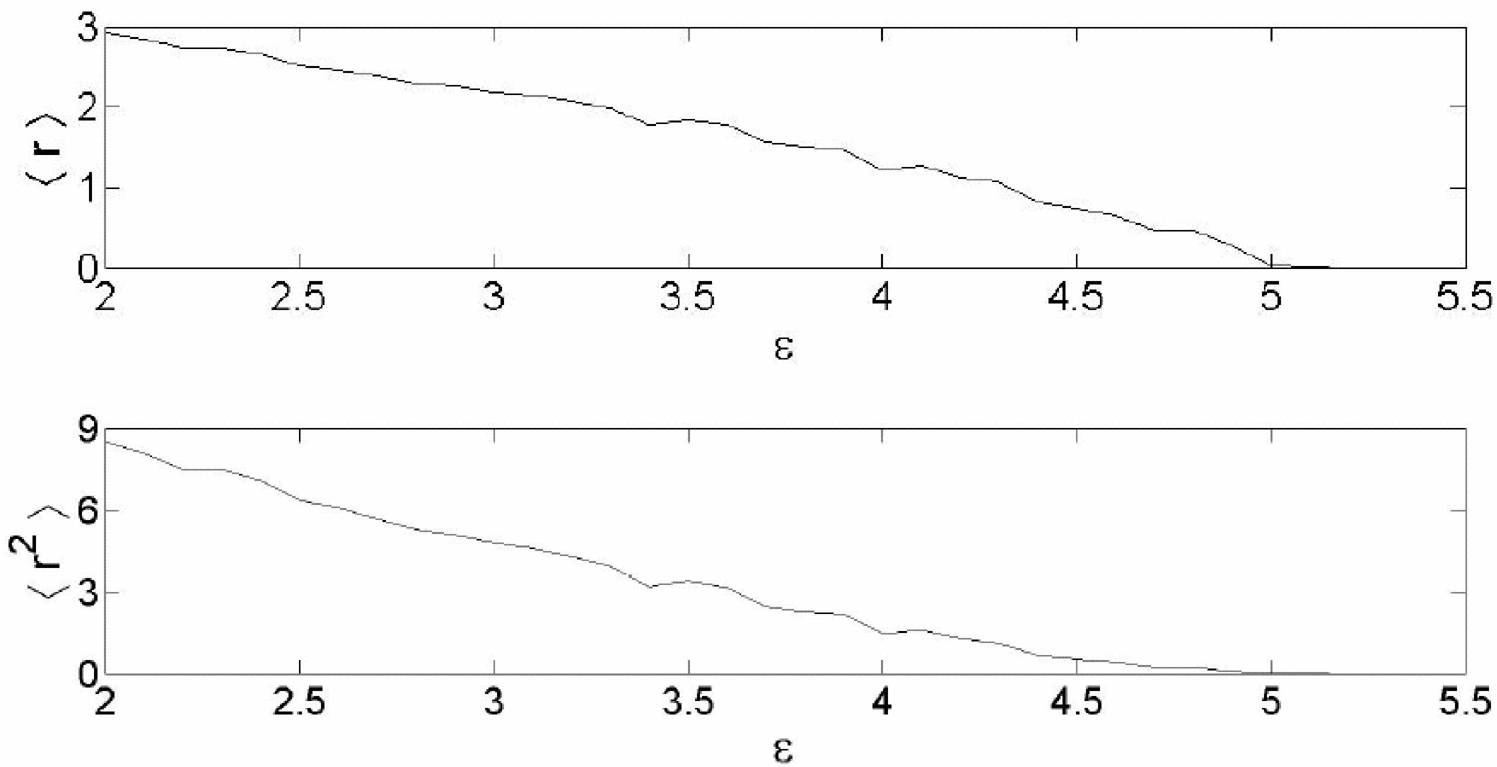



**Figure 4.5:** Moments $\langle r(t) \rangle$ and $\langle r(t)^2 \rangle$ as function of $\epsilon$

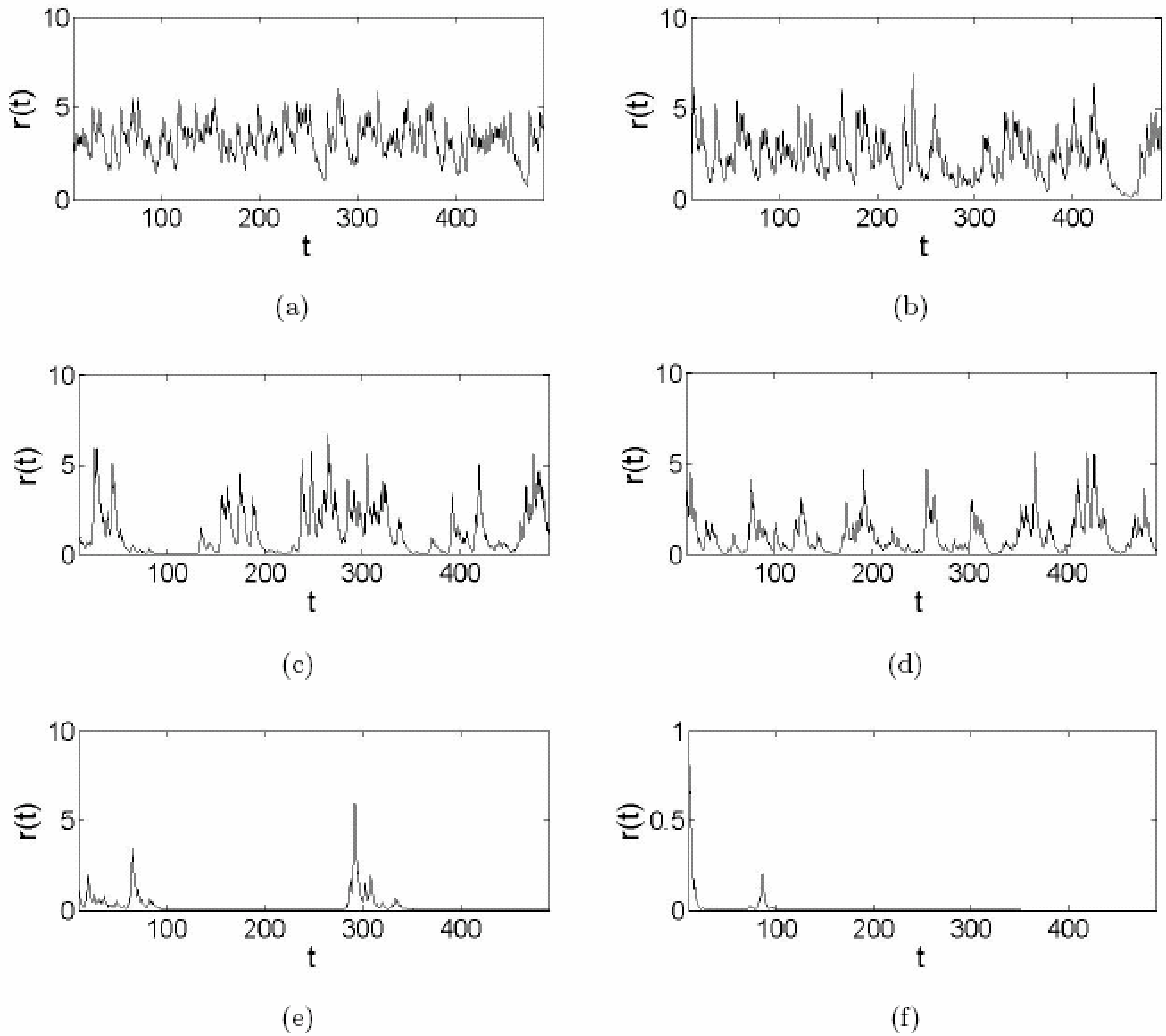



**Figure 4.2:** Time series of variable $r(t)$ for different values of $\epsilon$. (a) $\epsilon = 2$, (b) $\epsilon = 3$, (c) $\epsilon = 4$, (d) $\epsilon = 4.5$, (e) $\epsilon = 5$, (f) $\epsilon = 6$

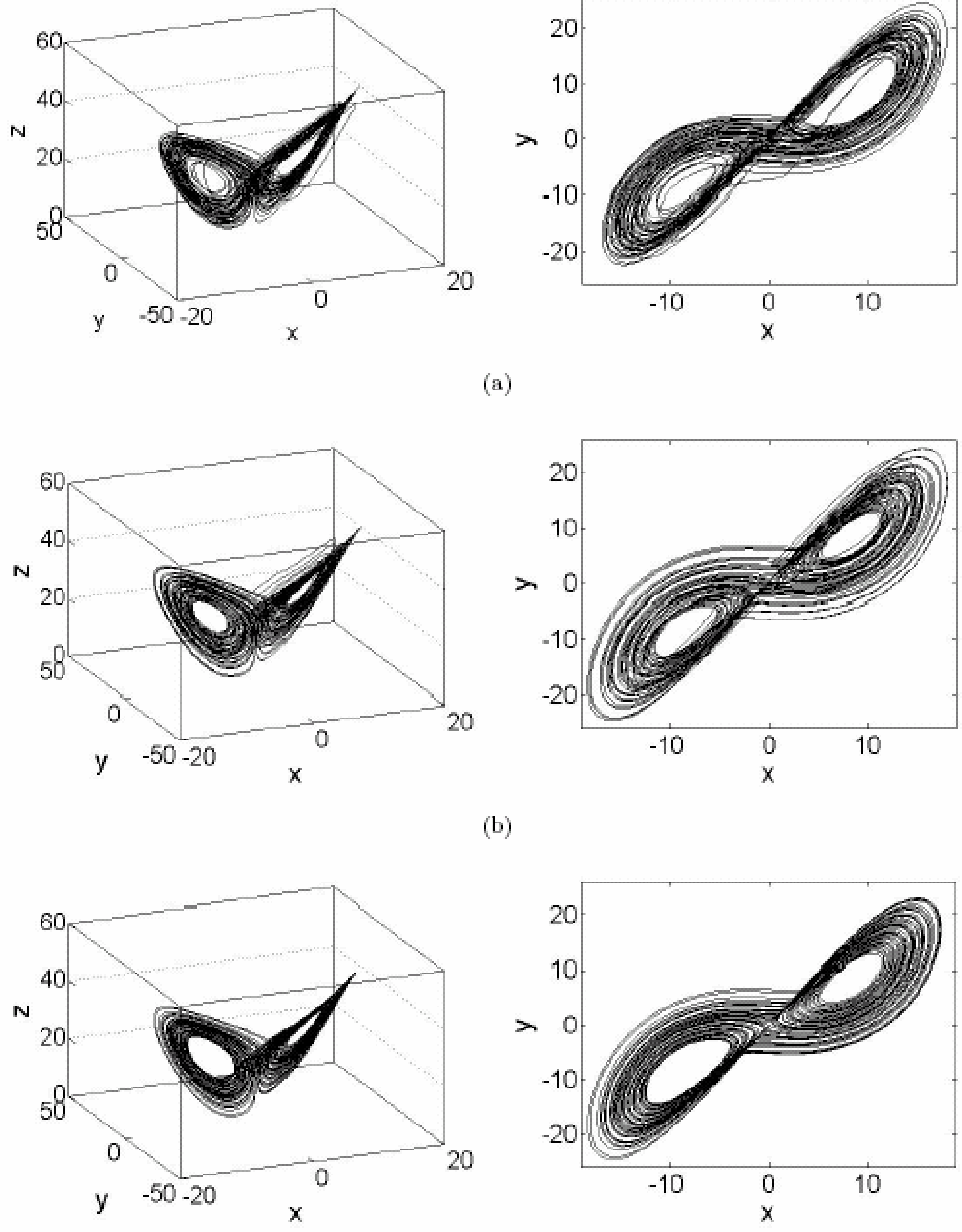



**Figure 4.3:** Projection of five dimensional phase space on the Lorenz subspace (first column) and projection onto the $x - y$ plane (second column), respectively. (a) $\epsilon = 2$, (b) $\epsilon = 4.5$, (c) $\epsilon = 7$

# Nonequilibrium phase transition in the ABCDE Model – numerical results

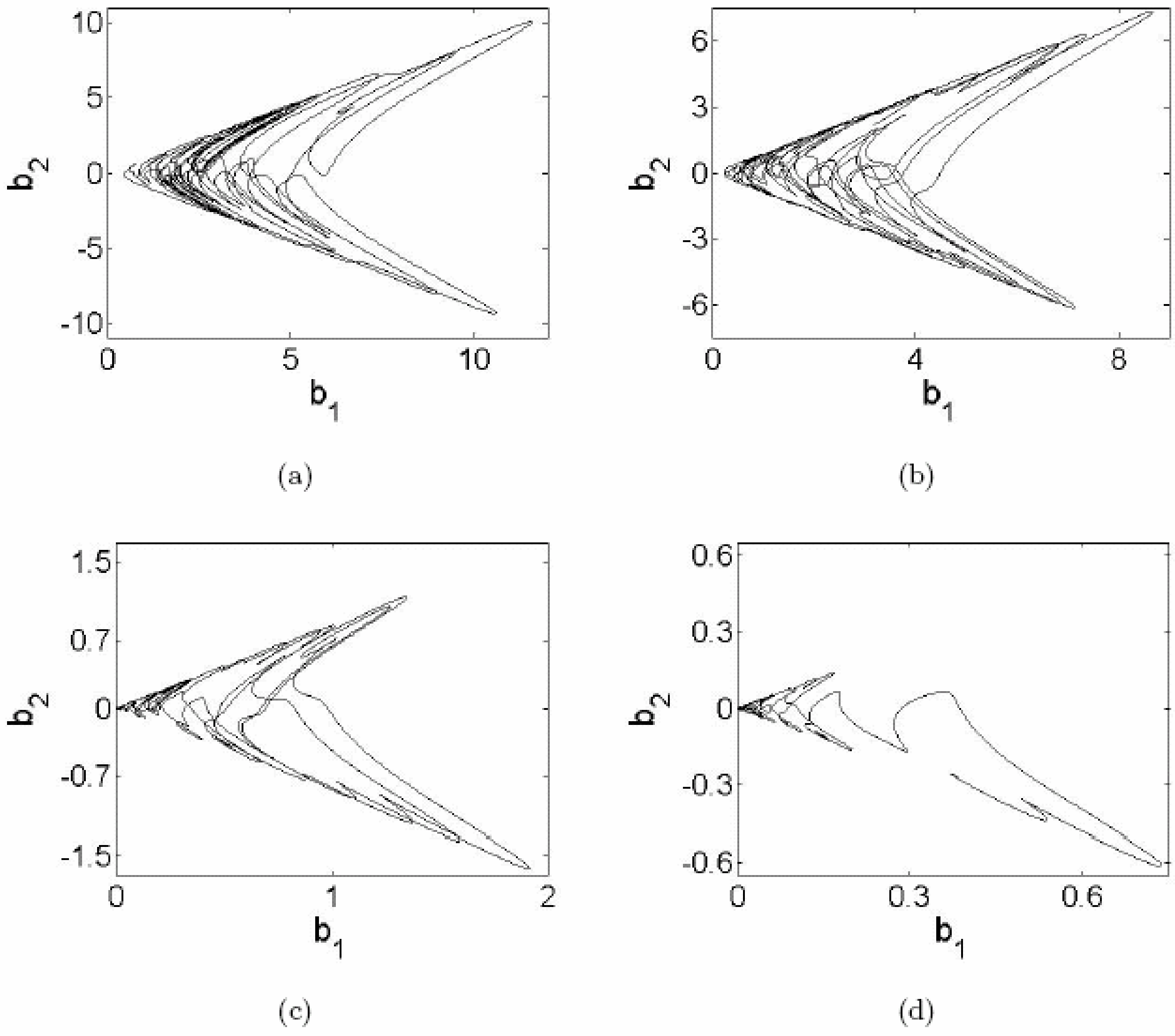
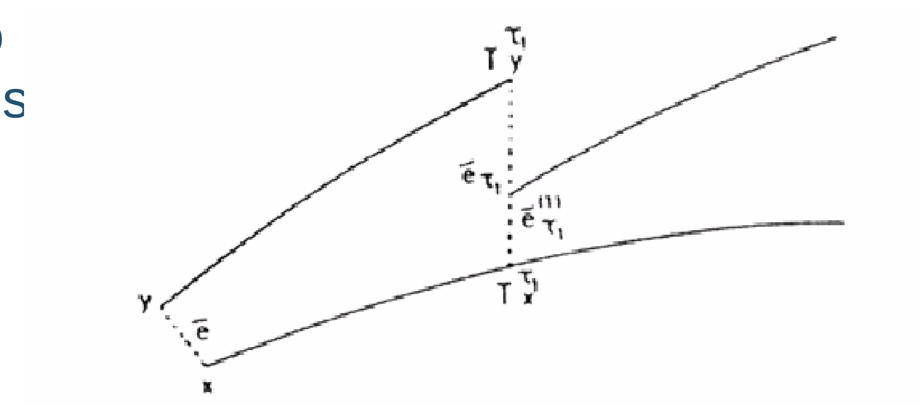


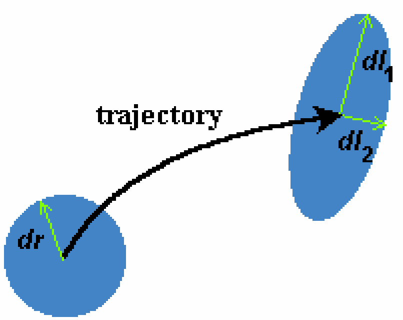

**Figure 4.4:** Projection of attractor onto the $b_1$-$b_2$ plane for (a) $\epsilon = 2$, (b) $\epsilon = 4.5$, (c) $\epsilon = 5.2$, (d) $\epsilon = 7$

# Nonequilibrium phase transition in the ABCDE Model – Lyapunov exponents

- Lyapunov characteristic exponents:
  Averaged rate of divergence (or convergence) of two neighbouring trajectories

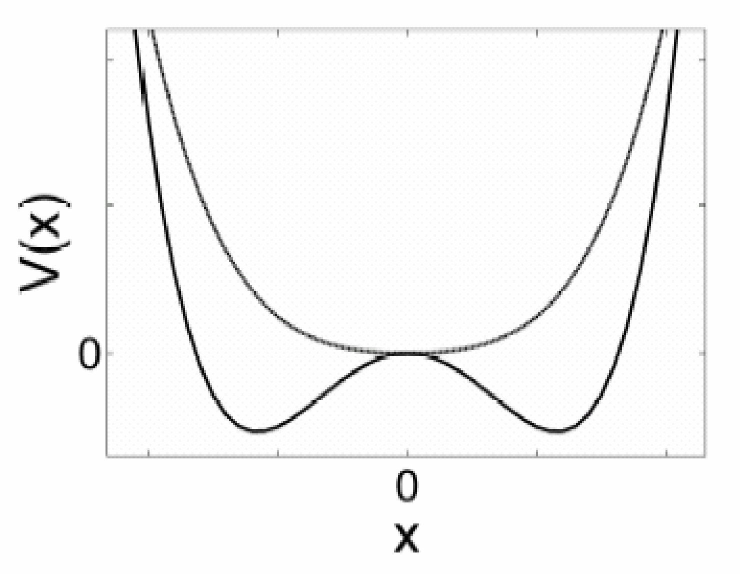


- Spectrum of Lyapunov characteristic exponents :




$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln\left(\frac{dl_i(t)}{dr}\right)$$

# Nonequilibrium phase transition in the ABCDE Model – Stability analysis

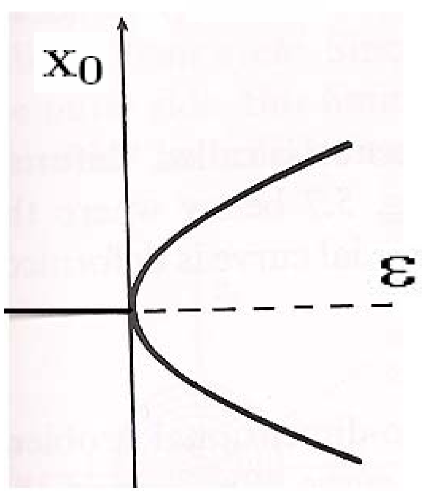



Model system: $$\dot{x} = \epsilon x - x^3$$

(sliding motion of ball in vase)

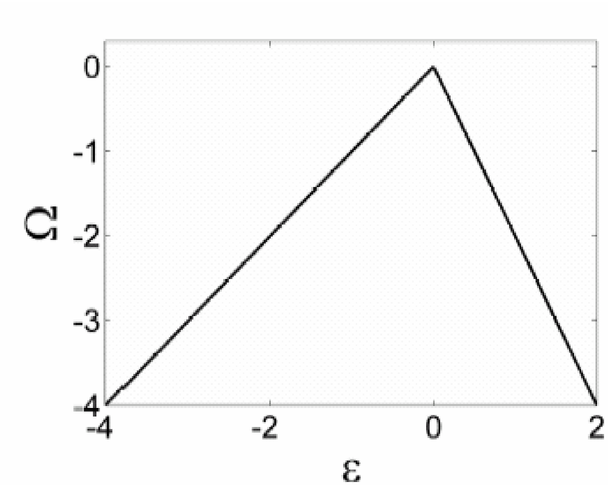
Stability analysis: $$x(t) = x_0 + \delta x(t)$$

Linearisation: $$\delta\dot{x} = \begin{cases} \epsilon\delta x & \epsilon < 0 \\ -2\epsilon\delta x & \epsilon > 0 \end{cases}$$




## Lyapunov exponent:

$$\Omega = \begin{cases} \epsilon & \epsilon < 0 \\ -2\epsilon & \epsilon > 0 \end{cases}$$



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$$\dot{b}_1 = -\epsilon a_1 b_1 + \alpha x b_2$$
$$\dot{b}_2 = -\epsilon a_2 b_2 + \alpha x b_1$$

$$\dot{x} = \sigma(-x + y) - b_1 b_2$$
$$\dot{y} = -y + (r - z)x$$
$$\dot{z} = -bz + xy$$

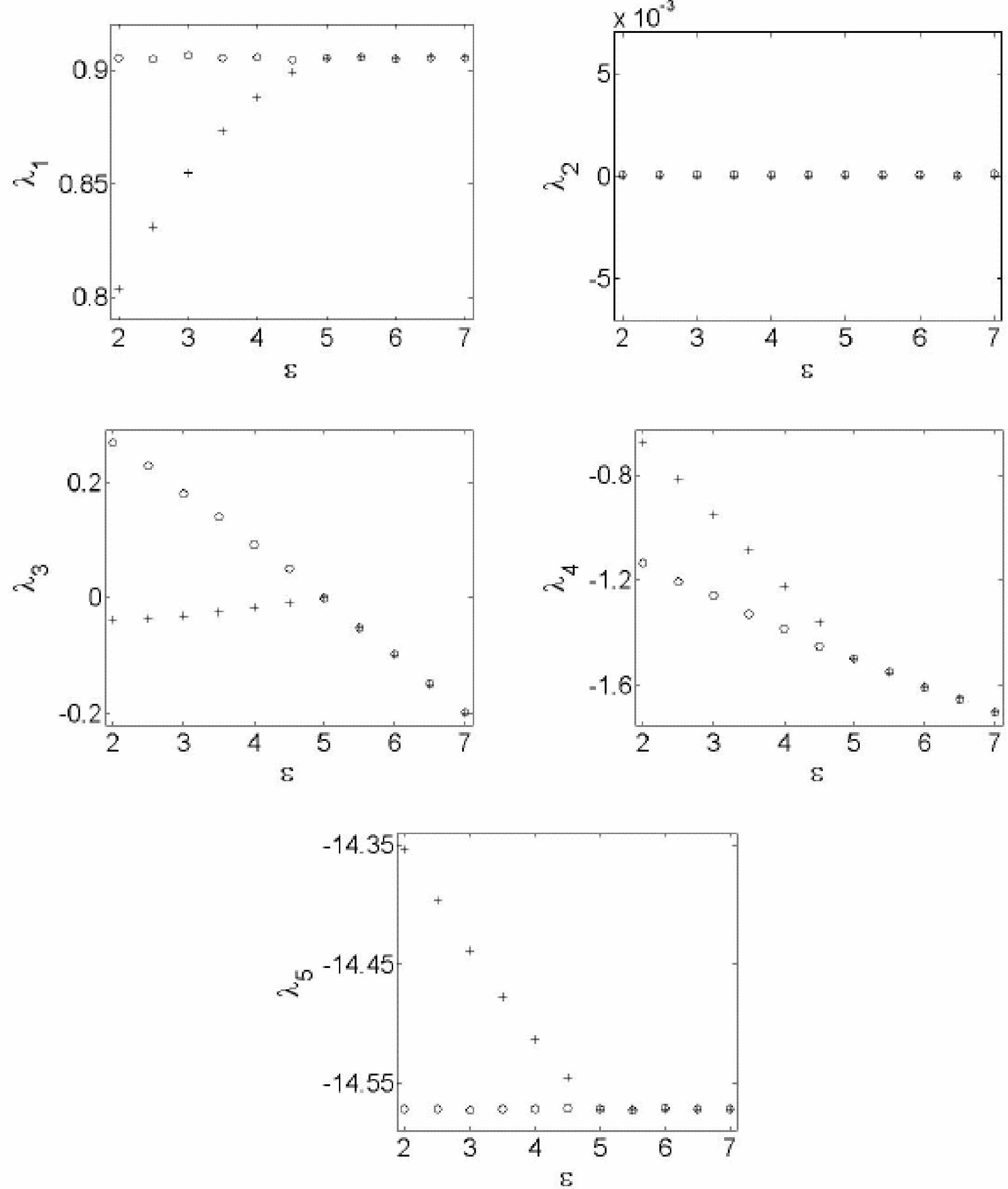
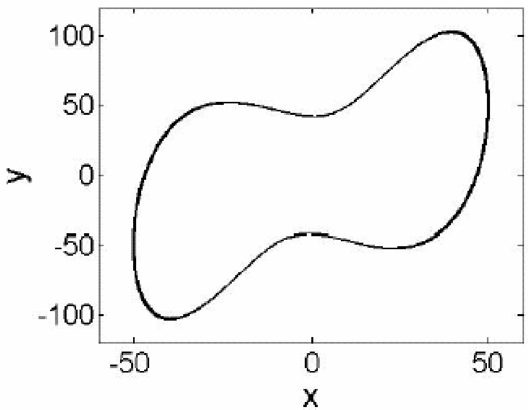


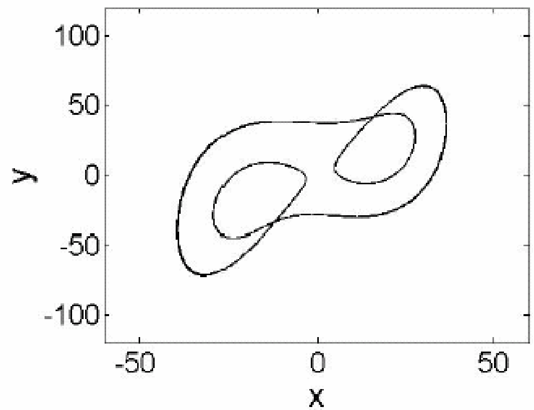

**Figure 6.3:** The five Lyapunov exponents in dependence of the order parameter $\epsilon$. **A.** o signals no back coupling of b onto x variables. **B.** (+) denotes result obtained with back coupling. Special attention is paid to the influence of the back reaction on the behavior of $\lambda_3$

# Lyapunov characteristic exponents

## Special case: limit cycle




(a) $R = 350$




(b) $R = 148.5$

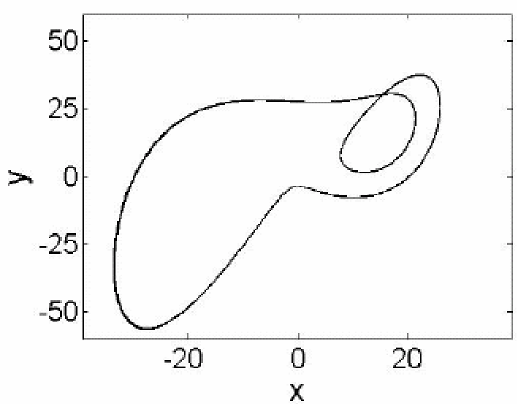
$$\dot{b}_1 = -\epsilon a_1 b_1 + \alpha x b_2$$

$$\dot{b}_2 = -\epsilon a_2 b_2 + \alpha x b_1$$

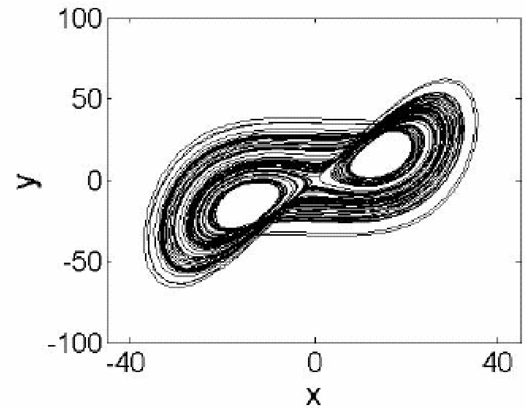
$$\dot{x} = \sigma(-x + y) - b_1 b_2$$

$$\dot{y} = -y + (r - z)x$$

$$\dot{z} = -bz + xy$$




(a) $R = 100.5$ NBC




(b) $R = 100.5$ WBC

$$d_t \mathbf{b} = \mathcal{A}(t)\mathbf{b}$$

$$\mathcal{A}(t) = \begin{pmatrix} -\epsilon a_1 & \alpha x \\ \alpha x & -\epsilon a_2 \end{pmatrix}$$

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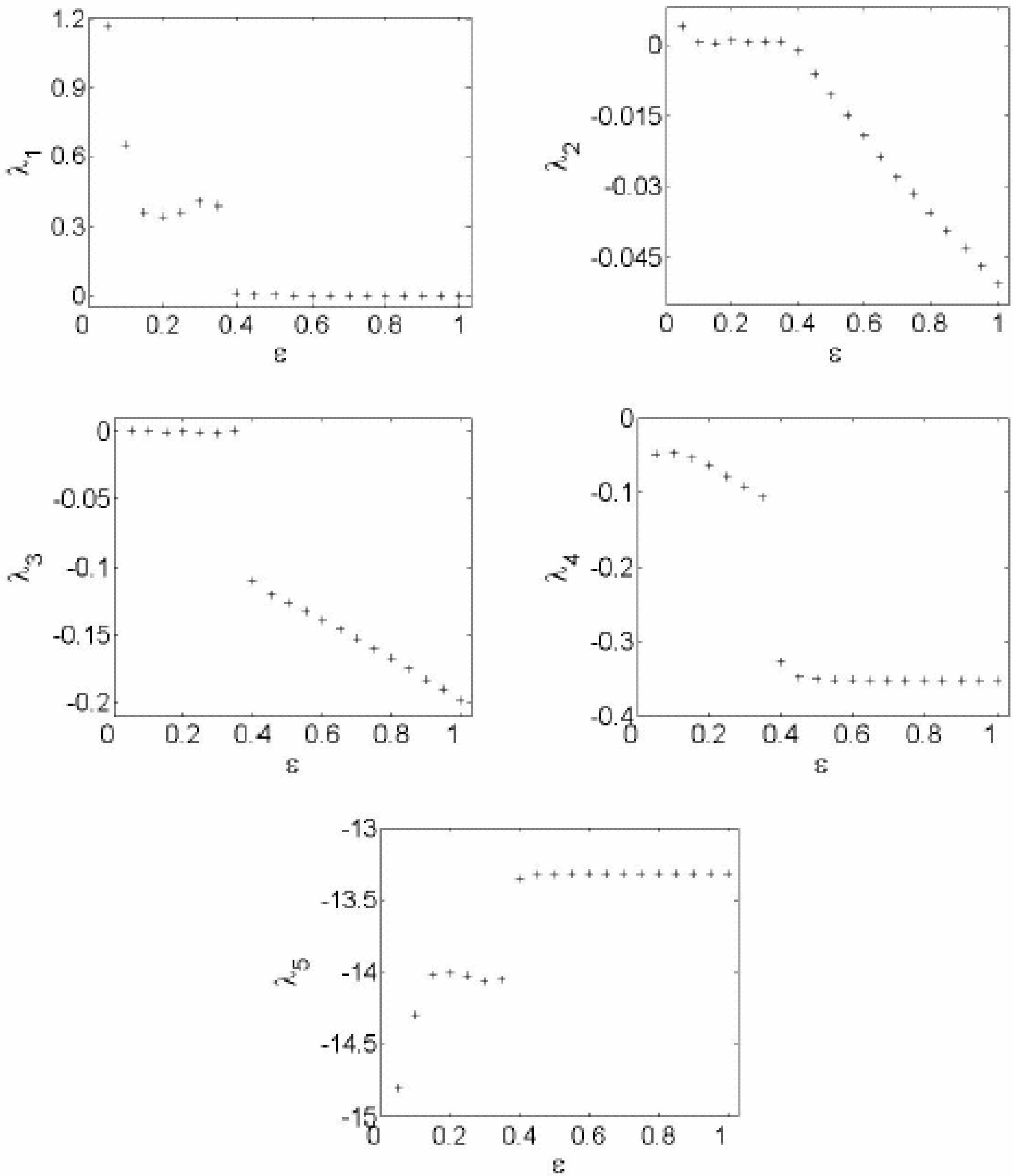


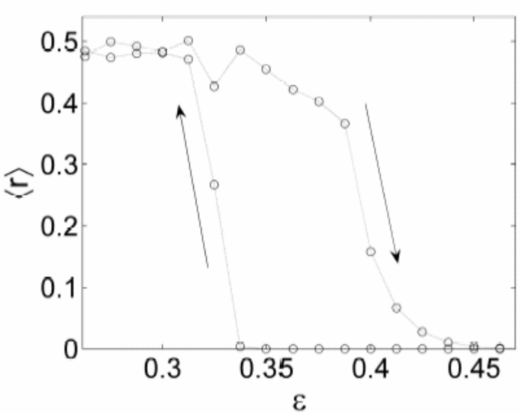


**Figure 6.6:** Lyapunov exponents in case of limit cycle ($r_h = 147.5$) as function of control parameter $\epsilon$

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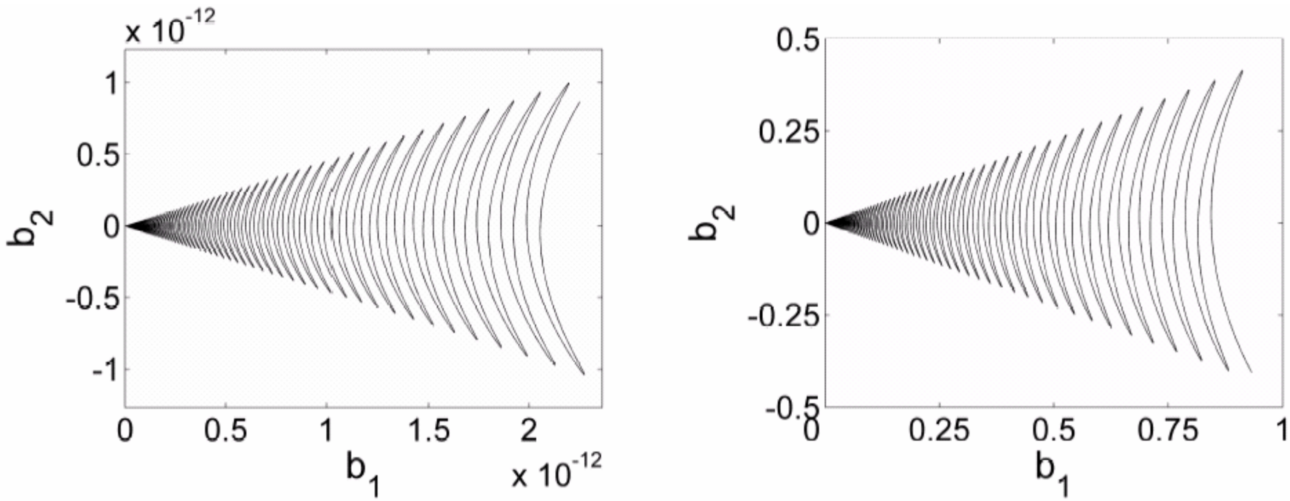
# Construction of solution for **b** variables employing Floquet's theory



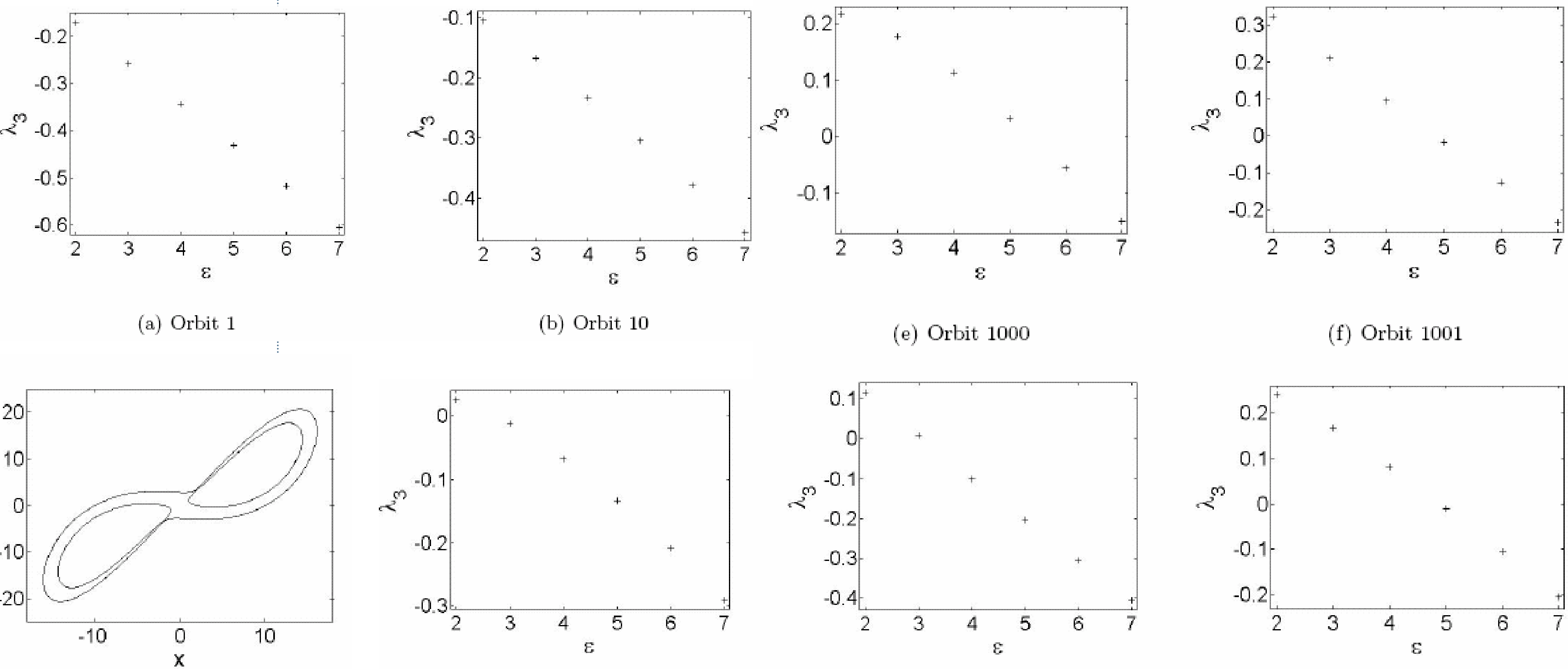

(a) Original (b) Fit

$$\mathbf{b}_f(t) = \mathbf{q}^{1,1}(t)\exp(\ln(\sigma_1)t/T) + \mathbf{q}^{1,2}(t)\exp(\ln(\sigma_2)t/T) \quad ,$$

whereby

$$\mathbf{q}^{1,1}(t) = \begin{pmatrix} c_1(\sin^2(2\pi t/T)+1) \\ c_2\sin(2\pi t/T) \end{pmatrix} \qquad \mathbf{q}^{1,2}(t) = \begin{pmatrix} c_3(\sin^2(2\pi t/T)+1) \\ c_4\sin(2\pi t/T) \end{pmatrix}$$

# Lyapunov exponents of periodic orbits (chaotic regime) under use of ordinary parameter values

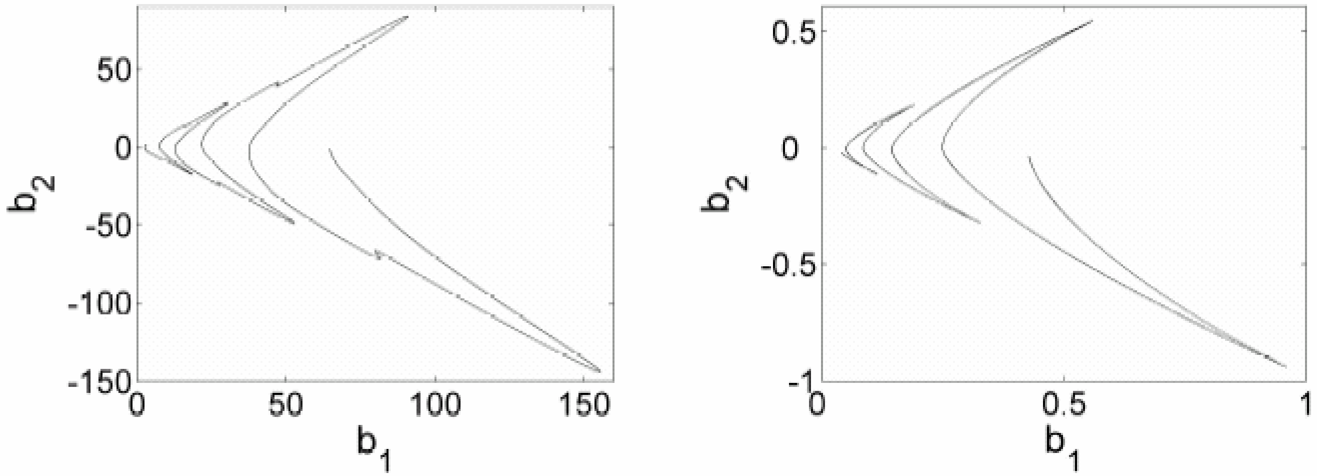



(a) Orbit 1 (b) Orbit 10 (e) Orbit 1000 (f) Orbit 1001

Orbit 10 (c) Orbit 100 (d) Orbit 101 (g) Orbit 1011

**Figure 6.9:** The Lyapunov exponent $\lambda_3$ for periodic orbits up to period $n = 4$ in dependence of the order parameter $\epsilon$

# Construction of solution for **b** variables employing Floquet‘s theory




(a) Original phase portrait

(b) Fit

**Figure 6.12:** Construction of solution for Orbit 1000 while $\epsilon = 2$

$$\mathbf{b}_f(t) = \mathbf{q}^{1,1}(t)\exp(\ln(\sigma_1)t/T) + \mathbf{q}^{1,2}(t)\exp(\ln(\sigma_2)t/T) \quad ,$$

whereas

$$\mathbf{q}^{1,1}(t) = \begin{pmatrix} c_1|\sin(2\pi t/T)|^{3/2} + c_2 \\ c_3\sin(2\pi t/T) \end{pmatrix} \qquad \mathbf{q}^{1,2}(t) = \begin{pmatrix} c_4|\sin(2\pi t/T)|^{3/2} + c_5 \\ c_6\sin(2\pi t/T) \end{pmatrix}$$

# Alternative approach to determine the critical characteristic exponent

Structure of equations implies variable transformation: $w(t) = \dfrac{b_2(t)}{b_1(t)}$

Substitution leads to two new expressions: $\dot{b}_1 = (-\epsilon a_1 + \alpha x w) b_1$

$$\dot{w} = -\epsilon(a_2 - a_1)w - (\alpha x)w^2 + \alpha x$$

Formal solution and corresponding characteristic exponent read:

$$b_1(t) = b_1(0) \exp\left( \int_{t_0}^{t} d\tau \left[-\epsilon a_1 + \alpha x(\tau) w(\tau)\right] \right)$$

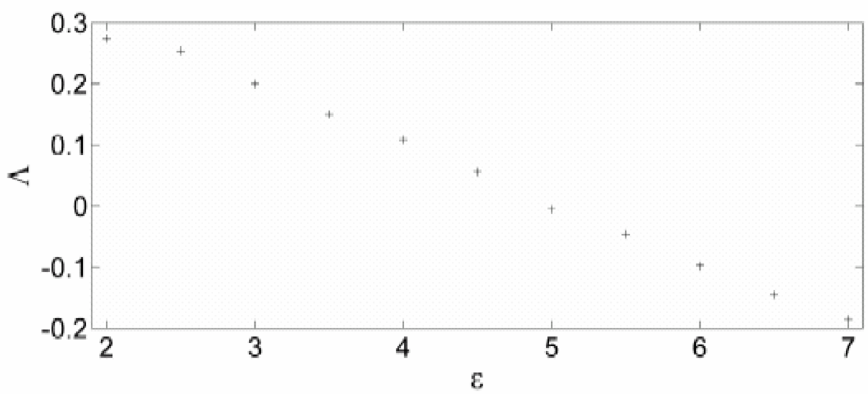



Figure 6.13: Characteristic exponent as function of order parameter $\epsilon$

$$b_1(t) = b_1(0) \exp\left(\Lambda(t)\, t\right)$$

$$\Lambda(t) = \frac{1}{t} \int_{t_0}^{t} d\tau \left[-\epsilon a_1 + \alpha x(\tau) w(\tau)\right]$$

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# Explicit temporal evolution of critical characteristic exponent

Origin of instability?

$$\Lambda(t) = \frac{1}{t}\int_{t_0}^{t} d\tau\,[-\epsilon a_1 + \alpha x(\tau) w(\tau)]$$

Define function according to:

$$\sigma(z) = \begin{cases} 1 & z > 0 \\ 0 & \text{else} \end{cases}$$

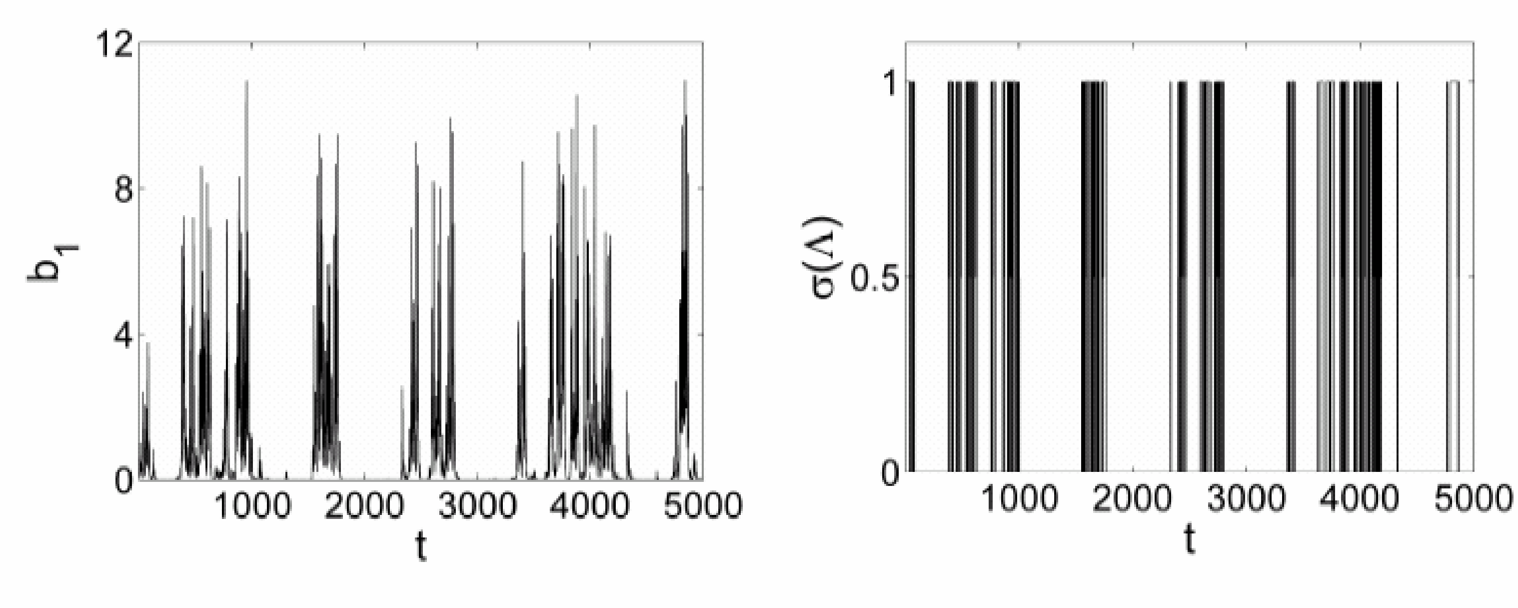



Figure 6.14: Time series and function $\sigma(\Lambda)$

# Stochastic treatment of transition in terms of Langevin equation

$$\frac{d\mathrm{x}_i(t)}{dt} = h_i(\mathbf{x}(t)) + \sum_j g_{i,j}(\mathbf{x}(t))\,\Gamma_j(t) \quad , \qquad i = 1, \ldots, n \quad ,$$

where

$$\langle \Gamma_i(t) \rangle = 0 \quad , \qquad \langle \Gamma_i(t)\Gamma_j(t') \rangle = Q\delta_{ij}\delta(t - t') \qquad \forall i, j \quad .$$

## Corresponding Fokker-Planck equation reads:

$$\frac{\partial p(\mathbf{x}, t)}{\partial t} = \left( -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} D_i^{(1)}(\mathbf{x}, t) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} D_{ij}^{(2)}(\mathbf{x}, t) \right) p(\mathbf{x}, t)$$

Interrelation between both representations:

$$D_i^{(1)}(\mathbf{x}, t) = h_i(\mathbf{x}, t) \quad ,$$

$$D_{ij}^{(2)}(\mathbf{x}, t) = Q \sum_k g_{ik}(\mathbf{x}, t) g_{jk}(\mathbf{x}, t)$$

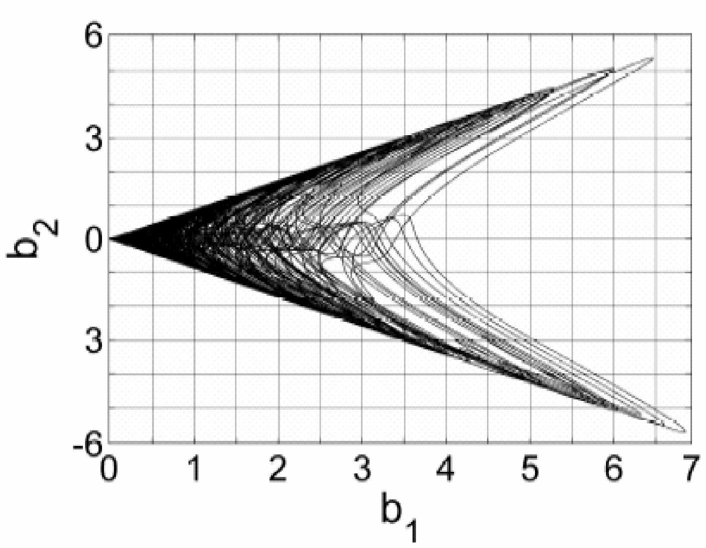
Numerical calculation according to stochastic definition:

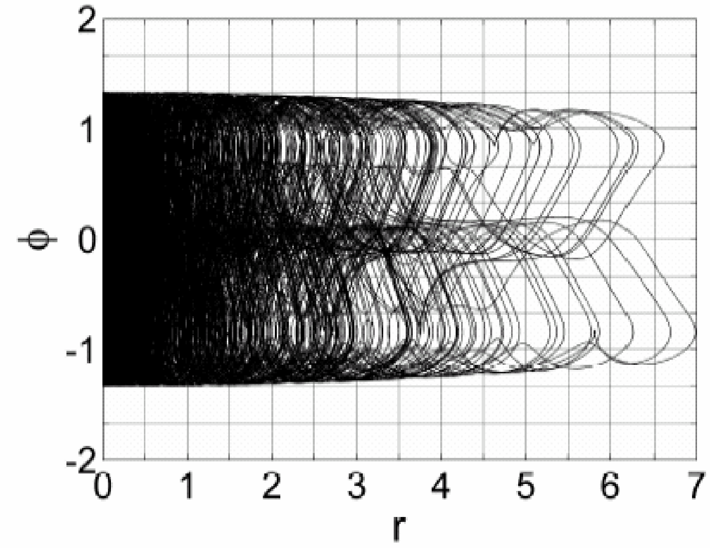
$$D_i^{(1)}(\mathbf{x}, t) = \lim_{\tau \to 0} \frac{1}{\tau} \langle x_i(t + \tau) - x_i \rangle_{\mathbf{x}(t) = \mathbf{x}} \quad ,$$

$$D_{ij}^{(2)}(\mathbf{x}, t) = \lim_{\tau \to 0} \frac{1}{\tau} \langle (x_i(t + \tau) - x_i)(x_j(t + \tau) - x_j) \rangle_{\mathbf{x}(t) = \mathbf{x}}$$

# Stochastic treatment of transition in terms of Langevin equation

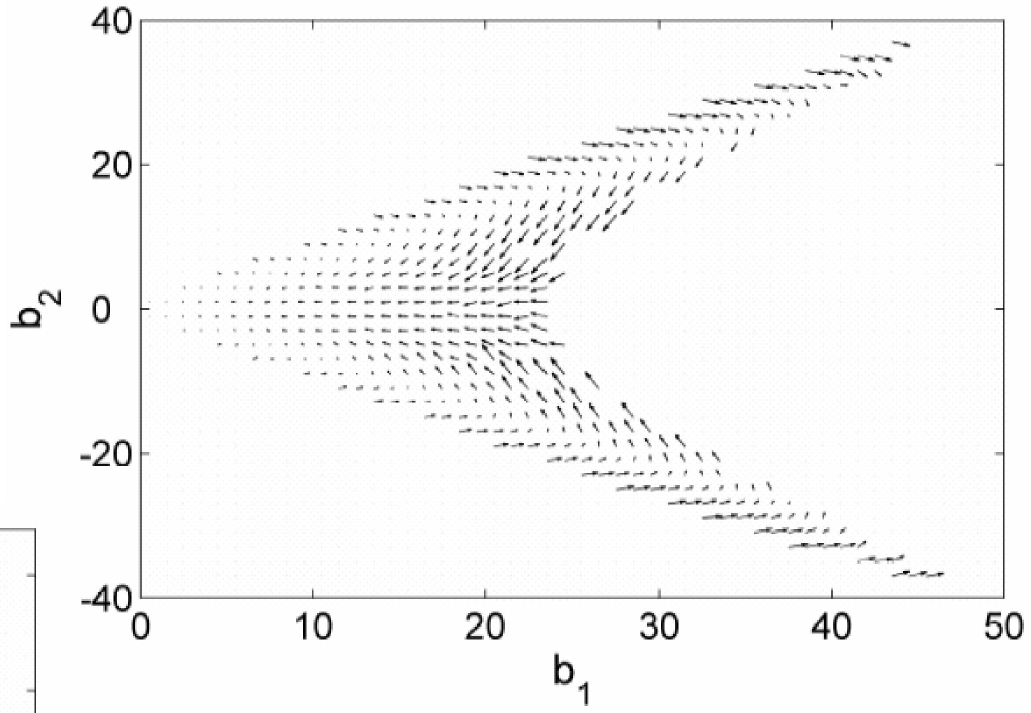
- Allocation of bins, choosing suitable bin size
- Computation of stationary drift and diffusion coefficients
- Setup of stochastic process to model deterministic dynamics

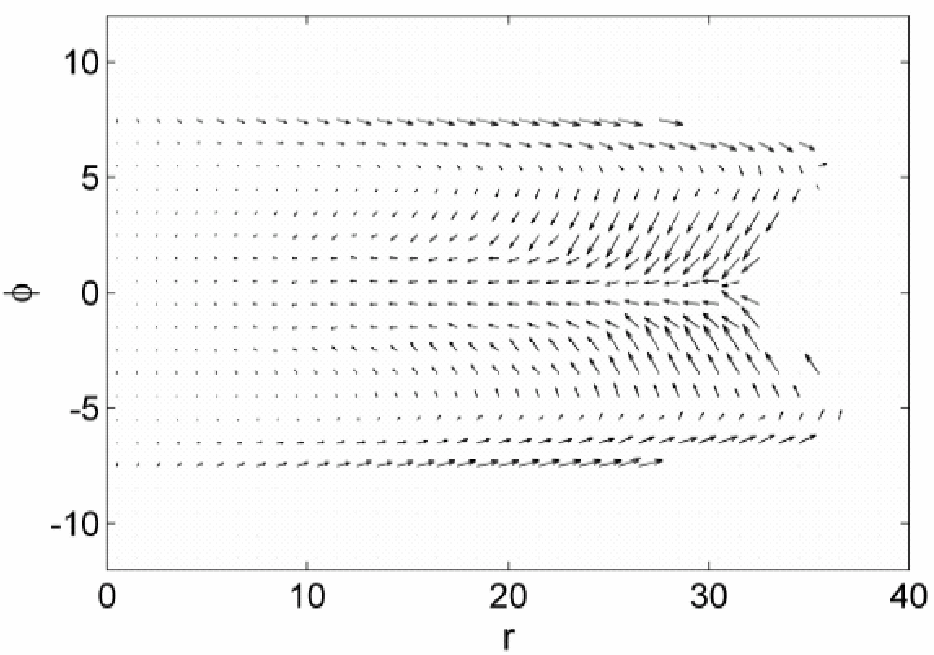



- Hyperbolic coordinates turn out to be more suitable for stochastic description

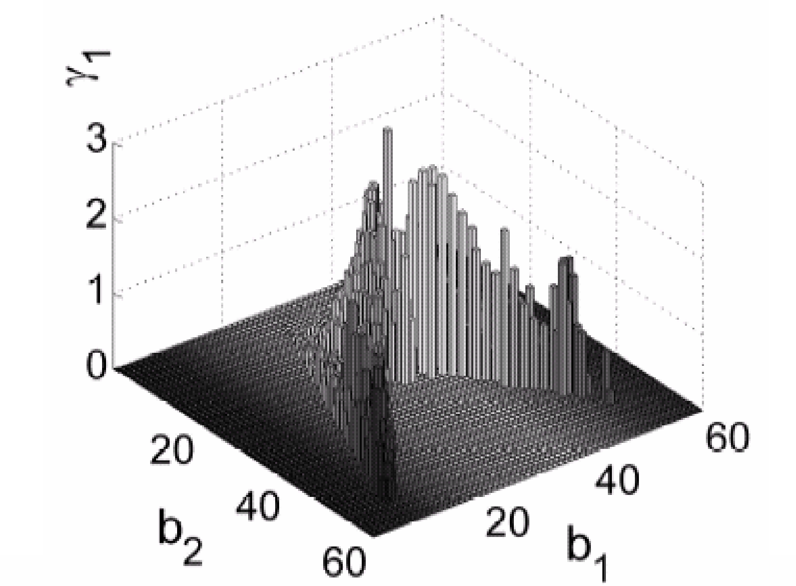
# Vector field representing drift coefficients $D^{(1)}$

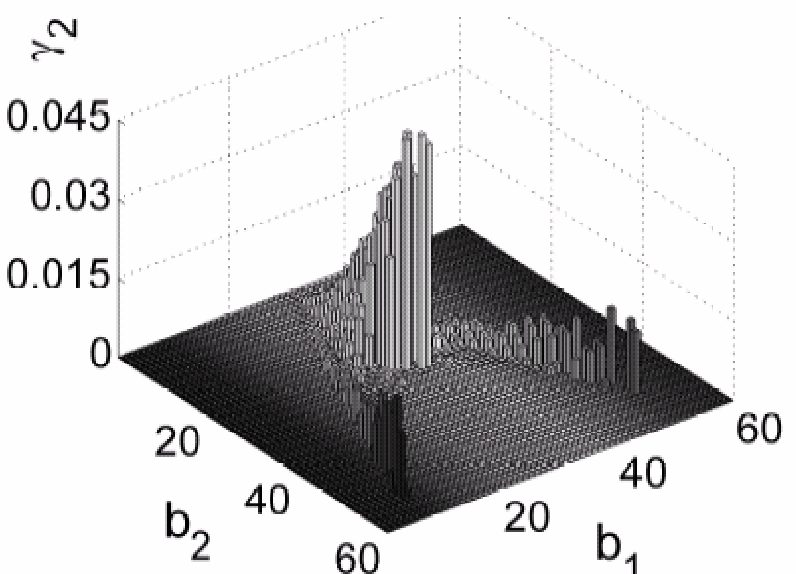


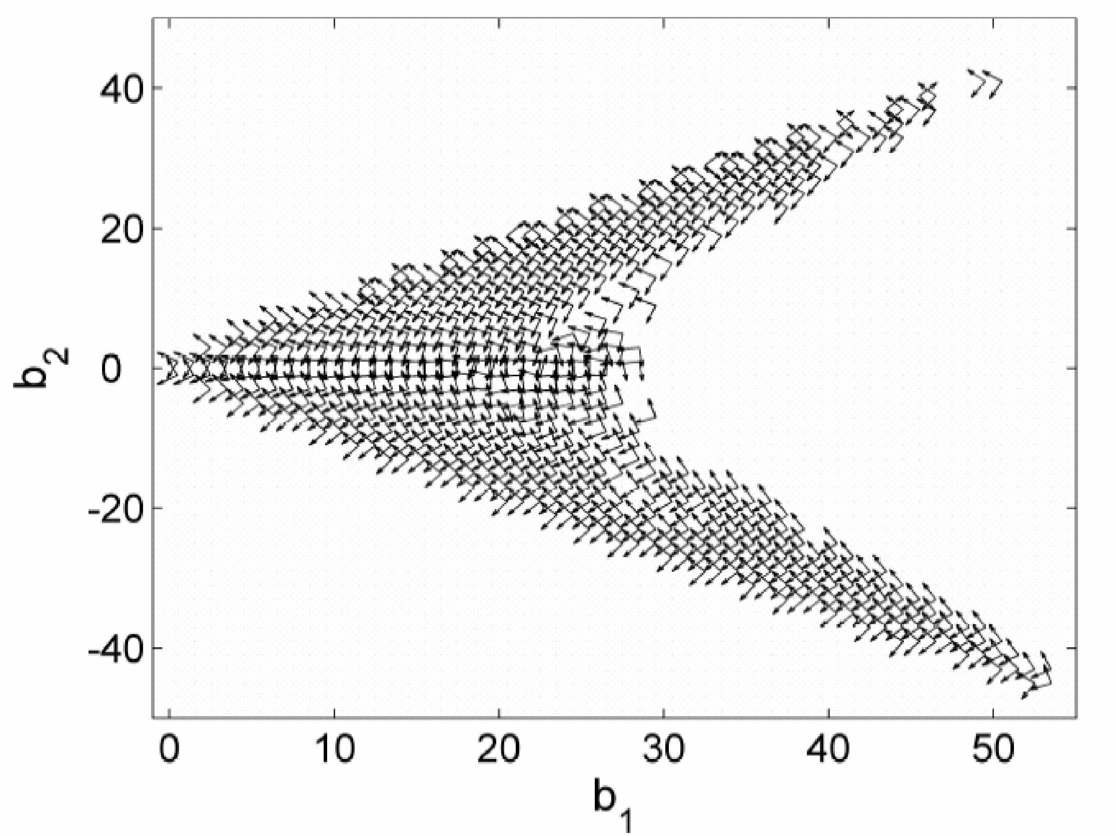




Drift coefficients define basic deterministic dynamics, fluctuations are guided by diffusion coefficients

# Eigenvectors and eigenvalues as characterization of diffusion coefficients $D^{(2)}$



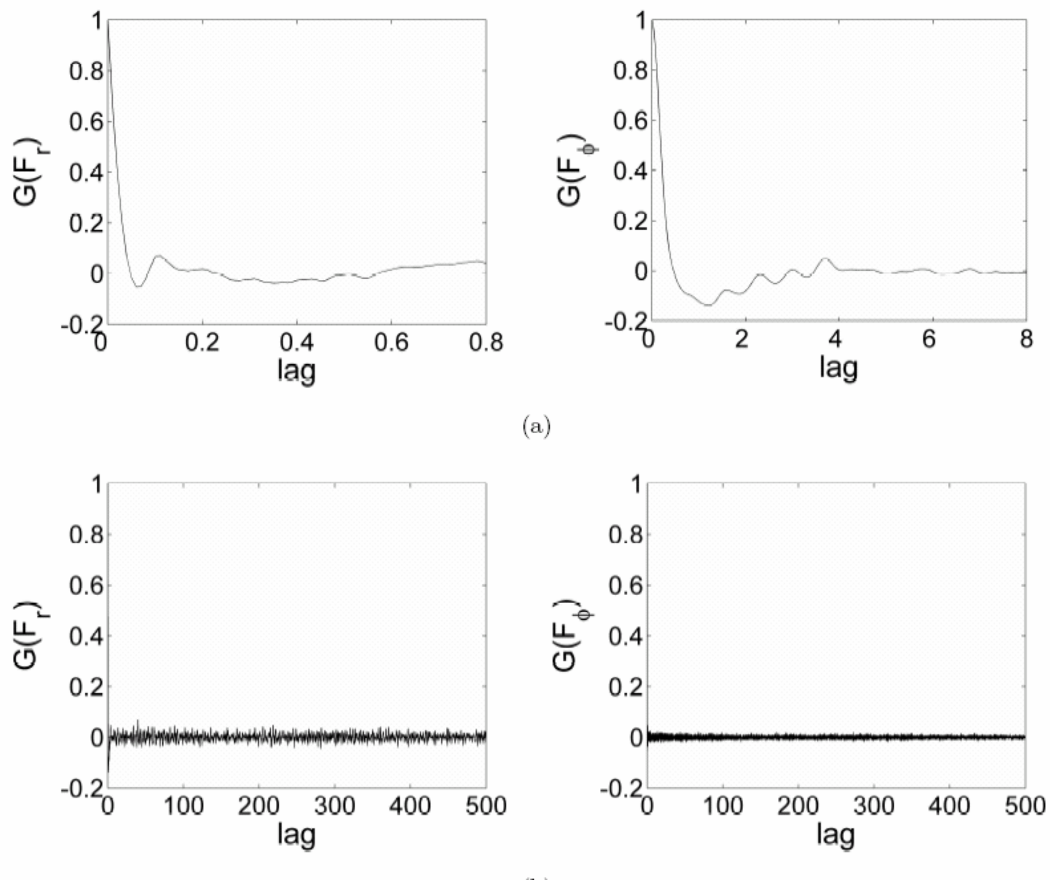



$$\frac{dx_i(t)}{dt} = h_i\left(\mathbf{x}(t)\right) + \sum_j g_{i,j}\left(\mathbf{x}(t)\right)\Gamma_j(t), \qquad i = 1, \ldots, n$$

# Investigation of characteristic force F

$$\mathbf{F} = \mathbf{b}(t + \Delta t) - \mathbf{b}(t) - \Delta t \mathbf{D}^{(1)}$$

Examination of autocorrelation function of F as measure of "stochasticity"



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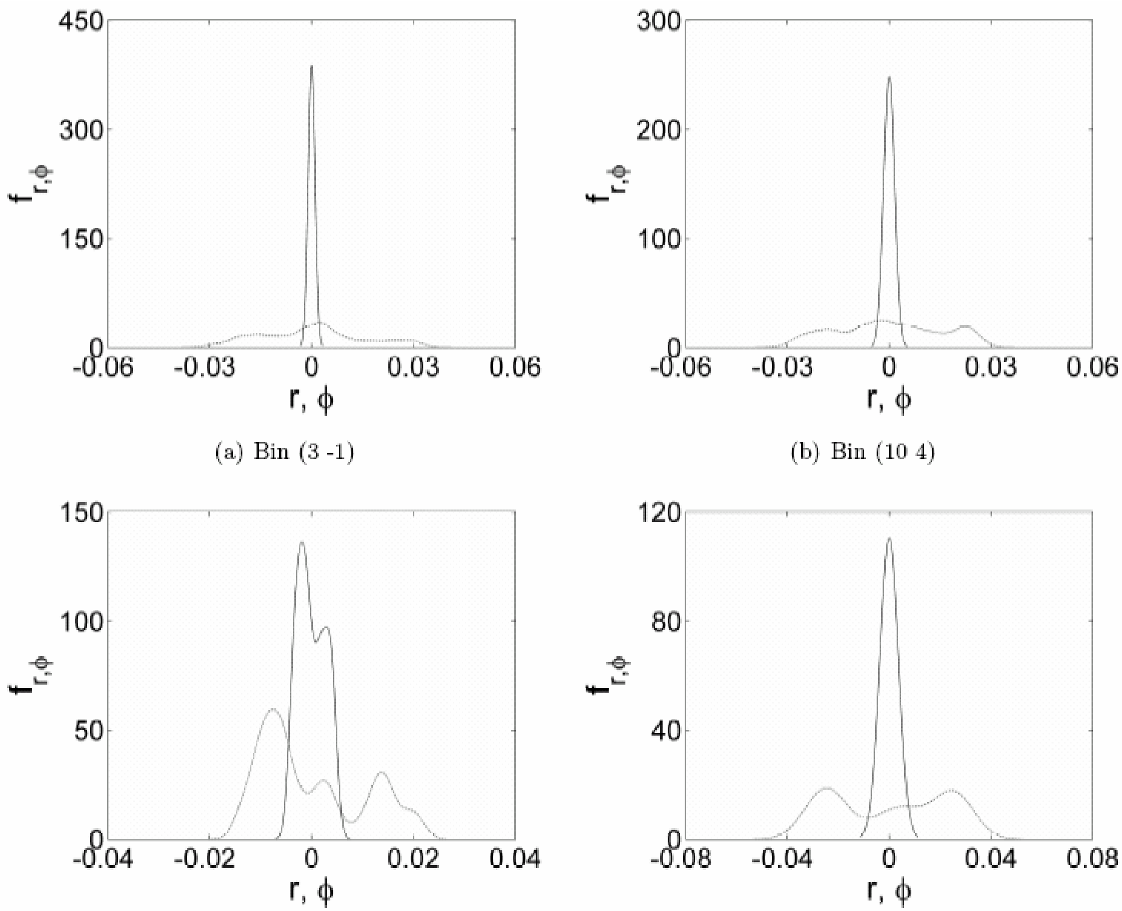
$$\frac{dx_i(t)}{dt} = h_i\left(\mathbf{x}(t)\right) + \sum_j g_{i,j}\left(\mathbf{x}(t)\right)\Gamma_j(t) \quad , \qquad i = 1, \ldots, n$$

# Investigation of characteristic force F

## What is an appropriate representation of the stochastic term ?




(a) Bin (3 -1)

(b) Bin (10 4)

(c) Bin (35 8)

(d) Bin (40 1)

**Figure 7.7:** r (-), $\phi(-.)$

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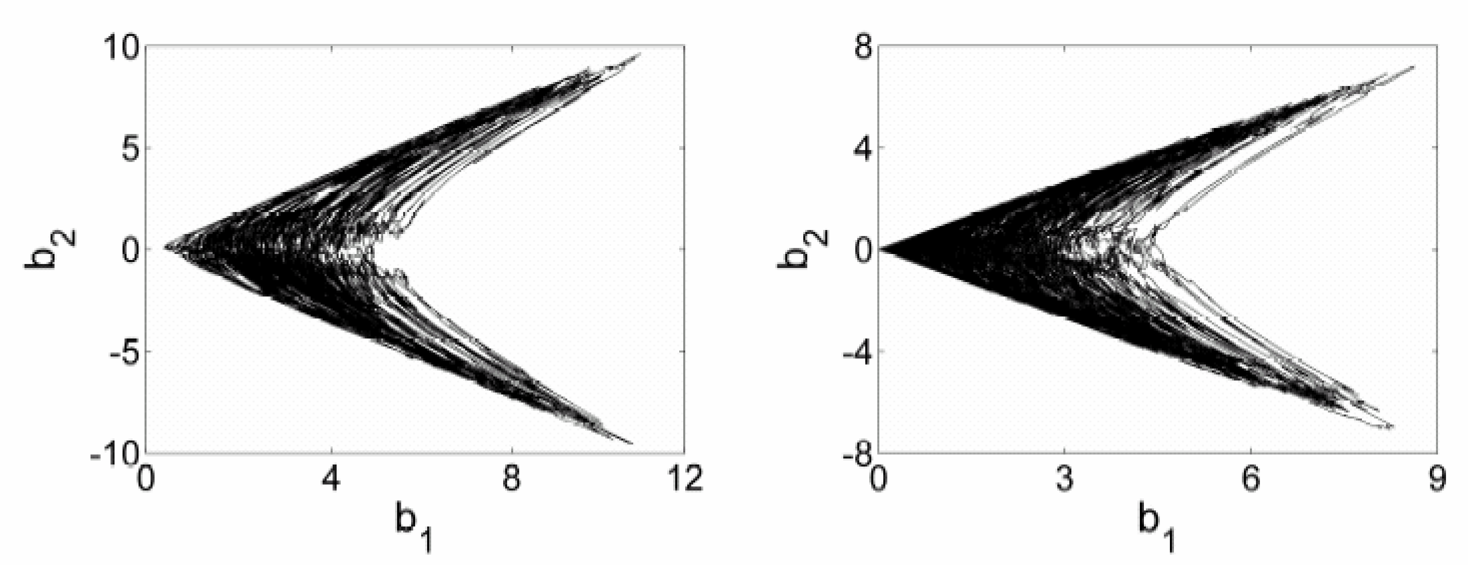
# Numerical results of the stochastic Langevin model

- Numerical integration of Langevin equation employing Gaussian distributed white noise process
- Phase space portrait for $\varepsilon = 2.5$ and $\varepsilon = 4.5$



# Numerical results of the stochastic Langevin model

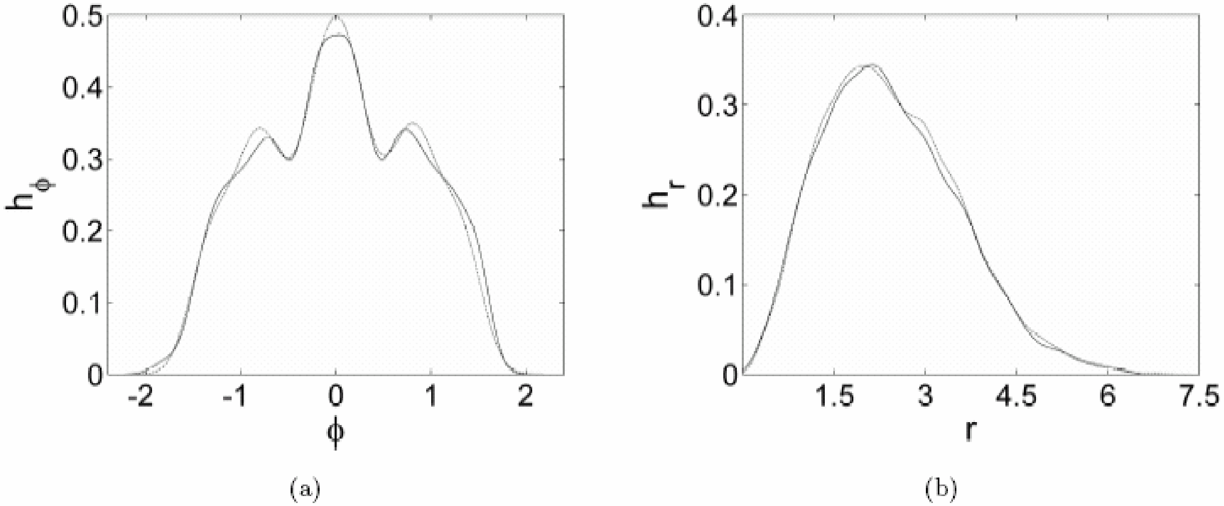



Figure 7.10: $\epsilon = 2.5$, Estimate (-), Data (-.)

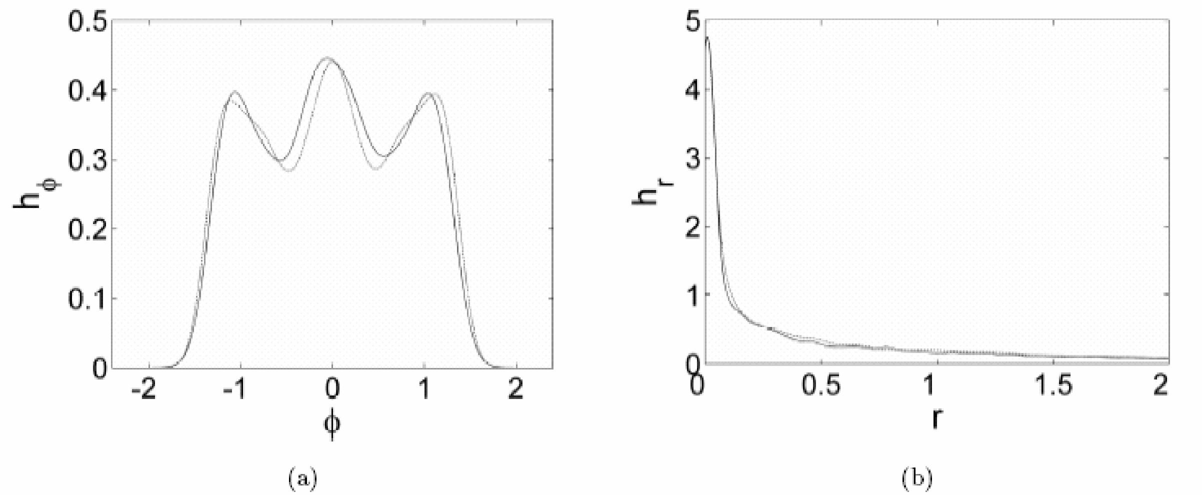



Figure 7.9: $\epsilon = 4.5$, Estimate (-), Data (-.)

Probability density estimates for both hyperbolic variables obtained from direct numerical integration of ABCDE equations and stochastic Langevin model

# Questions?