Chapter 2
Burgers Equation

One of the major challenges in the field of complex systems is a thorough understanding of the phenomenon of turbulence. Direct numerical simulations (DNS) have substantially contributed to our understanding of the disordered flow phenomena inevitably arising at high Reynolds numbers. However, a successful theory of turbulence is still lacking which would allow to predict features of technologically important phenomena like turbulent mixing, turbulent convection, and turbulent combustion on the basis of the fundamental fluid dynamical equations. This is due to the fact that already the evolution equation for the simplest fluids, which are the so-called Newtonian incompressible fluids, have to take into account nonlinear as well as nonlocal properties:

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t)+\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t) &= -\nabla p(\mathbf{x},t)+\nu\Delta\mathbf{u}(\mathbf{x},t)\,, \\ \nabla\cdot\mathbf{u}(\mathbf{x},t) &= 0\,. \end{aligned} \tag{2.1}$$

Nonlinearity stems from the convective term and the pressure term, whereas nonlocality enters due to the pressure term. Due to incompressibility, the pressure is defined by a Poisson equation

$$\Delta p(\mathbf{x},t) = -\nabla\cdot\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)\,. \tag{2.2}$$

In 1939 the dutch scientist J.M. Burgers [2] simplified the Navier-Stokes equation (2.1) by just dropping the pressure term. In contrast to Eq. (2.1), this equation can be investigated in one spatial dimension (Physicists like to denote this as 1+1 dimensional problem in order to stress that there is one spatial and one temporal coordinate):

$$\frac{\partial}{\partial t}u(x,t)+u(x,t)\frac{\partial}{\partial x}u(x,t) = \nu\frac{\partial^2}{\partial x^2}u(x,t)+F(x,t) \tag{2.3}$$

Note that usually the Burgers equation is considered without external force $F(x,t)$. However, we shall include this external force field.

The Burgers equation 2.3 is nonlinear and one expects to find phenomena similar to turbulence. However, as it has been shown by Hopf [13] and Cole [4], the homogeneous Burgers equation lacks the most important property attributed to turbulence: The solutions do not exhibit chaotic features like sensitivity with respect to initial conditions. This can explicitly shown using the *Hopf-Cole transformation* which transforms Burgers equation into a linear parabolic equation. From the numerical point of view, however, this is of importance since it allows one to compare numerically obtained solutions of the nonlinear equation with the exact one. This comparison is important to investigate the quality of the applied numerical schemes. Furthermore, the equation has still interesting applications in physics and astrophysics. We will briefly mention some of them.

Growth of interfaces: Deposition models

The Burgers equation (2.3) is equivalent to the so-called *Kardar-Parisi-Zhang (KPZ-) equation* which is a model for a solid surface growing by vapor deposition, or, the opposite case, erosion of material from a solid surface. The location of the surface is described in terms of a height function $h(\mathbf{x},t)$. This height evolves in time according to the KPZ-equation

$$\frac{\partial}{\partial t} h(\mathbf{x},t) - \frac{1}{2} \left(\nabla h(\mathbf{x},t)\right)^2 = \nu \frac{\partial^2}{\partial x^2} h(x,t) + F(x,t)\,. \tag{2.4}$$

This equation is obtained from the simple advection equation for a surface at $z = h(\mathbf{x},t)$ moving with velocity $\mathbf{U}(\mathbf{x},t)$

$$\frac{\partial}{\partial t} h(\mathbf{x},t) + \mathbf{U} \cdot \nabla h(\mathbf{x},t) = 0\,. \tag{2.5}$$

The velocity is assumed to be proportional to the gradient of $h(\mathbf{x},t)$, i.e. the surface evolves in the direction of its gradient. Surface diffusion is described by the diffusion term.

The Burgers equation (2.3) is obtained from the KPZ-equation just by forming the gradient of $h(\mathbf{x},t)$:

$$\mathbf{u}(\mathbf{x},t) = -\nabla h(\mathbf{x},t)\,. \tag{2.6}$$

2.1 Hopf-Cole Transformation

The Hopf-Cole transformation is a transformation, which maps the solution of the Burgers equation (2.3) to the heat equation

$$\frac{\partial}{\partial t} \psi(\mathbf{x},t) = \nu \Delta \psi(\mathbf{x},t)\,. \tag{2.7}$$

We perform the ansatz

$$\psi(\mathbf{x},t) = e^{h(\mathbf{x},t)/2\nu} \tag{2.8}$$

and determine

$$\Delta\psi = \frac{1}{2\nu}\left[\Delta h + \frac{1}{2\nu}(\nabla h)^2\right]e^{h/2\nu} \tag{2.9}$$

leading to

$$\frac{\partial}{\partial t}h - \frac{1}{2}(\nabla h)^2 = \nu\Delta h\,. \tag{2.10}$$

However, this is exactly the Kardar-Parisi-Zhang equation (2.4). The complete transformation is then obtained by combining

$$\mathbf{u}(\mathbf{x},t) = -\frac{1}{2\nu}\nabla \ln\psi(\mathbf{x},t)\,. \tag{2.11}$$

We explicitly see that the Hopf-Cole transformation turns the nonlinear Burgers equation into the linear heat conduction equation. Since the heat conduction equation is explicitly solvable in terms of the so-called heat kernel we obtain a general solution of the Burgers equation. Before we construct this general solution, we want to emphasize that the Hopf-Cole transformation applied to the multi-dimensional Burgers equation only leads to the general solution provided the initial condition $\mathbf{u}(\mathbf{x},0)$ is a gradient field. For general initial conditions, especially for initial fields with $\nabla\times\mathbf{u}(\mathbf{x},t)$, the solution can not be constructed using the Hopf-Cole transformation and, consequently, is not known in analytical terms. In one dimension spatial dimension it is not necessary to distinguish between these two cases.

2.2 General Solution of the 1D Burgers Equation

We are now in the position to formulate the general solution of the Burgers equation (2.3) in one spatial dimension with initial condition

$$u(x,0)\,, \qquad \psi(x,0) = e^{-\frac{1}{2\nu}\int^x dx' u(x',0)}\,. \tag{2.12}$$

The solution of the 1D heat equation can be expressed by the heat-kernel

$$\psi(x,t) = \int dx' G(x-x',t)\psi(x',0) \tag{2.13}$$

with the kernel

$$G(x-x',t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-x')^2}{4\nu t}}\,. \tag{2.14}$$

In terms of the initial condition (2.12) the solution explicitly reads

$$\psi(x,t) = \frac{1}{\sqrt{4\pi t}}\int dx' e^{-\frac{(x-x')^2}{4\nu t}-\frac{1}{2\nu}\int^{x'} dx'' u(x'',0)}\,. \tag{2.15}$$

The n-dimensional solution of the Burgers equation (2.3) for initial fields, which are gradient fields, are obtained analogously:

$$\psi(x,t) = \frac{1}{(4\pi t)^{d/2}} \int d\mathbf{x}' e^{-\frac{(\mathbf{x}-\mathbf{x}')^2}{4\nu t} - \frac{1}{2\nu} \int^{\mathbf{x}'} d\mathbf{x}'' \cdot \mathbf{u}(\mathbf{x}'',0)} . \tag{2.16}$$

Agian, we see that the solution exist provided the integral is independent of the integration contour:

$$\int^{\mathbf{x}'} d\mathbf{x}'' \cdot \mathbf{u}(\mathbf{x}'',0) = h(\mathbf{x}',t) . \tag{2.17}$$

We can investigate the limiting case of vanishing viscosity, $\nu \to 0$. In the expression for $\psi(x,t)$, eq. (2.16), the integral is dominated by the minimum of the exponential function,

$$\min_{x'} \left[-\frac{(x-x')^2}{4\nu t} - \frac{1}{2\nu} \int^{x'} dx'' u(x'',0) \right] . \tag{2.18}$$

2.3 Forced Burgers Equation

The Hopf-Cole transformation can be applied to the forced Burgers equation. It is straightforward to show that this leads to the parabolic differential equation

$$\frac{\partial}{\partial t} \psi(x,t) = \nu \Delta \psi(\mathbf{x},t) - U(\mathbf{x},t) \psi(\mathbf{x},t) , \tag{2.19}$$

where the potential is related to the force

$$\mathbf{F}(\mathbf{x},t) = -\frac{1}{2\nu} \nabla U(\mathbf{x},t) . \tag{2.20}$$

The relationship with the Schrödinger equation for a particle moving in the potential $U(\mathbf{x},t)$ is obvious. Recently, the Burgers equation with a fluctuating force has been investigated [18]. Interestingly, Burgers equation with a linear force, i.e. a quadratic potential

$$U(x,t) = a(t) x^2 \tag{2.21}$$

for an arbitrary time dependent coefficient $a(t)$ could be solved analytically [9].

2.4 Numerical Treatment

Consider a real one-dimensional velocity field $u(x,t)$ which obeys the Burgers equation (2.3). According to the notation, presented in Sec. 1.3 the linear and nonlinear operators for this PDE take the form

$$L(u(x,t)) = \nu \frac{\partial^2}{\partial x^2} u(x,t) \tag{2.22}$$
$$N(u(x,t)) = -u(x,t) \frac{\partial}{\partial x} u(x,t)$$

ν denotes the kinematic viscosity. The Burgers equation is known to steepen negative gradients leading to the formation of so-called *shocks*. These shocks are smoothed out by viscous effects. For a sufficiently high viscosity, this equation may efficiently be solved by a pseudospectral method, as we want to exemplify in the following. Time-stepping can be achieved with a numerical scheme of your choice, fourth-order Runge-Kutta methods (see Appendix A) turn out to do a rather good job. After initializing an initial condition $u(x,t=0)$ the field is transformed to Fourier space yielding $\tilde{u}(k,t=0)$. The right hand side then may be treated in several easy steps according to

- dealias $\tilde{u}(k,t)$
- calculate the derivative according to $ik\tilde{u}(k,t)$
- transform $\tilde{u}(k,t)$ and $ik\tilde{u}(k,t)$ back to real space
- calculate the nonlinearity $N(x,t) = u(x,t) \cdot \frac{\partial}{\partial x} u(x,t)$
- transform $N(x,t)$ back to Fourier space
- dealias $\tilde{N}(k,t)$
- calculate the Laplacian $-k^2\tilde{u}(k,t)$
- build the output array according to $\tilde{N}(k,t) - \nu k^2 \tilde{u}(k,t)$

Taking for example a sinusoidal function as an initial conditions, negative gradients steepen yielding the characteristic shock structure.