Introduction to Quantum Field Theory
Problem sheet 3

Deadline: Wednesday 18 November 2015 (12 am)
at Dr. Giudice's office (KP 301) and Ms Sonja Esch (KP 310)

Topics covered: Parity operator, Dirac equation, Classical Field Theory.

1. (2 P) The parity operator P in Dirac theory is defined by the conditions:

$$P\gamma^0 P^{-1} = \gamma^0, \quad P\gamma^i P^{-1} = -\gamma^i \quad (i = 1, 2, 3), \quad P^4 = \mathbf{1}.$$

   This is solved by $P = \gamma^0$. There are other solutions for P: determine the most general one.

2. (2 P) Given the spin operator $\vec{S} = \frac{\hbar}{2}\vec{\Sigma}$, calculate $\vec{S}^2$ and determined the spin s from $\vec{S}^2 = \hbar^2 s(s+1)\mathbf{1}$.

3. (4 P) Calculate the commutators $[H, \vec{L}]$ and $[H, \vec{S}]$ and check that $[\vec{L} + \vec{S}, H] = 0$. H is the Dirac Hamiltonian operator.

4. (3 P) Derive the field equations and the Hamiltonian from the following Lagrangian for a one-dimensional elastic string:

$$\mathscr{L} = \frac{\mu}{2}\dot{\varphi}^2 - \frac{\kappa}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 - \lambda\varphi^4.$$

5. (4 P) Let $\varphi(x)$ be a complex scalar field with Lagrange density

$$\mathscr{L} = (\partial_\mu\varphi^*)(\partial^\mu\varphi) - m^2|\varphi|^2 - \frac{g}{6}|\varphi|^4.$$

   a) Consider φ and φ^* as two distinct fields and derive the field equations for them. Find the Hamiltonian.

   b) Let the two real fields φ_1 and φ_2 be defined through

$$\varphi = \frac{1}{\sqrt{2}}\left(\varphi_1 + \mathrm{i}\varphi_2\right).$$

   Write $\mathscr{L}$ in terms of φ_1 and φ_2. Derive the field equations for φ_1 and φ_2 and show that they are equivalent to those from a).