Econophysics WS1999/2000: Some Notes on Portfolio Theory

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1 Introduction

In this lecture we will revisit portfolio theory as pioneered by Markowitz [4]. The following notes mainly gives the basic formulæ without much explaining text. Great part of the material is from [1]. Many enlightening figures can be found in [2], which also is a classical reference to portfolio theory. For a short autobiography of Markowitz and some additional material on CAPM, lower partial moments, a relation to spin glasses, and references see also the corresponding webpage `http://pauli.uni-muenster.de/~lemm/econoWS99/portfolio.html`.

2 Portfolios with correlated assets

Following the discussion of optimal portfolios for uncorrelated assets in the last lecture, we study now portfolios of correlated assets.

Let again be M = number of assets, x_0 = value of riskless asset, x_i = value of risky asset i, $i \neq 0$, and n_i = number of assets of kind i.

Investing in a portfolio with initial wealth

$$W = \sum_{i=0}^{M} n_i x_i^0, \tag{1}$$

at some later time T the wealth of portfolio will be

$$S(T) = \sum_{i=0}^{M} n_i x_i(T). \tag{2}$$

For convenience, we take again $W = 1$, $x_i^0 = 1$, i.e., $p_i = n_i$ so that the condition $\sum_{i=0}^{M} p_i = 1$ must hold.

To be specific, we will now asssume the δx_i to be Gaussian random variables

$$p(\delta x_1, \cdots, \delta x_M) = \frac{1}{\sqrt{(2\pi)^M \det C}} e^{-\frac{1}{2}\sum_{ij}^{M}(\delta x_i - m_i) C_{ij}^{-1} (\delta x_j - m_i)}. \qquad (3)$$

The real symmetric, positive definite covariance matrix C can be diagonalized by an orthogonal transformation $O^T = O^{-1}$. Thus, the asset combinations e_i defined by

$$\delta x_i = m_i + \sum_j O_{ij} e_j, \qquad (4)$$

are uncorrelated.

In particular, we assume the expected returns of asset i

$$m_i = < \delta x_i >, \qquad (5)$$

to be known. Hence, the expected return of portfolio is the linear combination

$$m_p = \sum_{i=0}^{M} p_i m_i. \qquad (6)$$

Furthermore, we also assume the covariance matrix C_{ij} of assets to be known

$$\begin{aligned} C_{ij} &= < (\delta x_i - m_j)(\delta x_j - m_j) > & (7) \\ &= < \delta x_i \delta x_j > - < \delta x_i > m_j - m_i < \delta x_j > + m_i m_j & (8) \\ &= < \delta x_i \delta x_j > - m_i m_j. & (9) \end{aligned}$$

This is a nontrivial assumption, as guesses for covariances are difficult to obtain in practice. Similarly, we find for the covariance between asset i and the portfolio p

$$\begin{aligned} C_{ip} &= < (\delta x_i - m_j)(\delta S - m_p) > & (10) \\ &= \sum_{i=1}^{M} p_j < (\delta x_i - m_j)(\delta x_j - m_j) > & (11) \\ &= \sum_{i=1}^{M} C_{ij} p_j, & (12) \end{aligned}$$

and finally for the variance of portfolio p

$$D_p = C_{pp} = \sum_{i=1}^{M} p_i C_{ip} = \sum_{i,j=1}^{M} p_i C_{ij} p_j = \sum_{i=1}^{M} p_i^2 C_{ii} + \sum_{i,j=1, j\neq i}^{M} p_i C_{ij} p_j. \qquad (13)$$

The main idea of Markowitz was to find an optimal portfolio by minimizing its variance D_p fixing its expected return m_p. Minimizing D_p must be done under some constraints. Necessary constraints are

$$\text{“normalization”:} \qquad \sum_{i=0}^{M} p_i = 1, \tag{14}$$

$$\text{expected return:} \qquad \sum_{i=0}^{M} p_i m_i = m_p. \tag{15}$$

Note, that some p_i might be negative if short selling of assets is allowed. Other possible constraints, describing specific situations, can be, for example,

$$\text{dividends:} \qquad \sum_{i=0}^{M} p_i d_i = D, \tag{16}$$

$$\text{no short selling:} \qquad \sum_{i=0}^{M} |p_i| > 0, \tag{17}$$

$$\text{diversification:} \qquad \sum_{i=0}^{M} p_i^2 = C, \tag{18}$$

$$\text{margins:} \qquad \sum_{i=0}^{M} |p_i| x_i = F. \tag{19}$$

Example 1: Minimize portfolio variance D_p subject to (i) $\sum_{i=0}^{M} p_i = 1$ and (ii) $\sum_{i=0}^{M} p_i m_i = m_p$. Implementing the first constraint explicitly and introducing a Lagrange multiplier ζ for the second results in

$$E(p_0, p_1, \cdots, p_M) = \sum_{i,j=1}^{M} p_i C_{ij} p_j - \zeta \sum_{i=0} p_i m_i \tag{20}$$

$$= \sum_{i,j=1}^{M} p_i C_{ij} p_j - \zeta \sum_{i=1} p_i (m_i - m_0) - \zeta m_0. \tag{21}$$

It follows from $\partial E / \partial p_i = 0$ for $i \neq 0$

$$2 \sum_{i=1}^{M} C_{ij} p_j = \zeta (m_i - m_0) \quad \Big(= 2C_{ip} \Big), \tag{22}$$

and thus, inverting the covariance C_{ij}

$$p_i = \frac{\zeta}{2} \sum_{j=1}^{M} C_{ij}^{-1} (m_i - m_0). \tag{23}$$

The Lagrange multiplier ζ is obtained from

$$m_p = \sum_{i=0}^{M} p_i m_i \tag{24}$$

$$
\begin{aligned}
&= m_0 + \sum_{i=1}^{M} p_i(m_i - m_0) && (25) \\
&= m_0 + \frac{\zeta}{2} \sum_{i,j=1}^{M} (m_i - m_0) C_{ij}^{-1} (m_j - m_0), && (26)
\end{aligned}
$$

i.e.,

$$
\zeta = 2 \frac{m_p - m_0}{\sum_{i,j=1}^{M} (m_i - m_0) C_{ij}^{-1} (m_j - m_0)}. \tag{27}
$$

For D_p we find

$$
\begin{aligned}
D_p &= \sum_{i=1}^{M} p_i C_{ij} p_j && (28) \\
&= \frac{\zeta^2}{4} \sum_{i,j}^{M} (m_i - m_0) C_{ij}^{-1} (m_j - m_0) && (29) \\
&= \frac{(m_p - m_0)^2}{\sum_{i,j}^{M} (m_i - m_0) C_{ij}^{-1} (m_j - m_0)}, && (30)
\end{aligned}
$$

hence

$$
\zeta = 2 \frac{D_p}{m_p - m_0} \tag{31}
$$

Example 2: Minimize portfolio variance D_p without riskless asset subject to (i) $\sum_{i=0}^{M} p_i = 1$ and (ii) $\sum_{i=0}^{M} p_i m_i = m_p$. Implementing the both constraints by introducing Lagrange multiplier ζ and μ results in

$$
E(p_1, \cdots, p_M) = \sum_{i,j=1}^{M} p_i C_{ij} p_j - \zeta \sum_{i=1} p_i m_i - \mu \sum_{i=1} p_i. \tag{32}
$$

It follows from $\partial E / \partial p_i = 0$ for $i \neq 0$

$$
p_i = \sum_{j=1}^{M} C_{ij}^{-1} (\frac{\zeta}{2} m_i + \frac{\mu}{2}). \tag{33}
$$

The point with the lowest risk has $\xi = 0$, so that

$$
p_i = \frac{\mu}{2} \sum_{j=1}^{M} C_{ij}^{-1}, \tag{34}
$$

and for μ follows

$$
1 = \sum_i p_i = \frac{\mu}{2} \sum_{ij} C_{ij}^{-1}, \tag{35}
$$

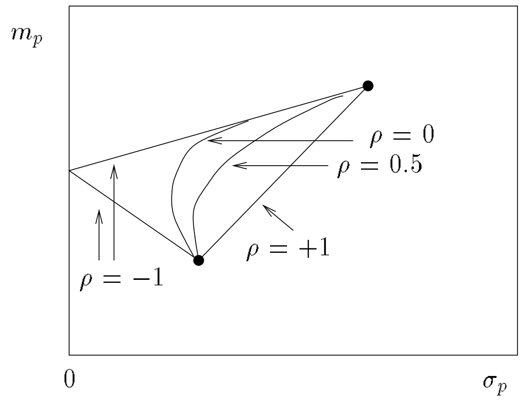

Figure 1: Portfolios of two risky correlated assets without short selling in the m_p–σ_p plane.

i.e.,

$$\mu = \frac{2}{\sum_{ij} C_{ij}^{-1}}. \tag{36}$$

All optimal portfolios must have more risk. For the general case the Lagrange multipliers xi and μ can be obtained similarly as in Example 1.

Fig. 1 shows the dependency of a portfolio of two risky assets from their correlation coefficient. The correlation coefficient, defined as $\rho = \frac{C_{12}}{\sqrt{C_{11}C_{22}}}. = \frac{C_{12}}{\sigma_1\sigma_2}$, can only take values between -1 (perfect anti–correlation) and $+1$ (perfect correlation). In the extreme case of perfect anti–correlation, i.e., $\rho = -1$, the two risky assets can be combined to a riskfree portfolio. If one of the two assets is riskfree, all the straight lines in the figure coincide (see example 1).

3 Capital Asset Pricing Model (CAPM)

Inserting Eq. (31) into Eq. (22) results in a relation between m_i and m_p,

$$m_i = m_0 + \underbrace{\frac{C_{ip}}{D_p}}_{\beta = C_{ip}/C_{pp}} (m_p - m_0). \tag{37}$$

(known as security market line) or, in terms of the standard deviation $\sigma_p = \sqrt{D_p}$,

$$m_i = \underbrace{m_0}_{\text{market price of time}} + \underbrace{\frac{C_{ip}}{\sigma_p}}_{\text{risk of asset}} \underbrace{\frac{m_p - m_0}{\sigma_p}}_{\text{market price of risk}}. \tag{38}$$

The Capital Asset Pricing Model (CAPM) assumes that all agents use this mean variance portfolio with the same guesses for m_i and D_i. It follows that the whole market can be considered as a mean variance portfolio, the so called market portfolio. Furthermore, the only free parameter of a rational investor should be the proportion p_0 of the riskless asset which is mixed with the market portfolio. I.e, letting m_p denote the expected return of the market portfolio, individual portfolios with returns m_i are on the line

$$m_i = m_0 + \sigma_i \frac{(m_p - m_0)}{\sigma_p}. \tag{39}$$

(capital market line) parametrized by σ_i [2]. Much empirical work has been devoted to check the validity of the CAPM with differing results.

Linear regression

Equation (37) follows also from predicting m_i from m_p by a linear regression model. Specifically, assume n data pairs (y_j, x_j). In the case of the CAPM these data will be pairs $(m_i(t_j), m_p(t_j))$, sampled for example at times t_j. Out aim is to find an optimal prediction of the y_i based on the x_i (i.e.,, for CAPM, prediction of the $m_i(t_j)$ based on the $m_p(t_j)$). To determine an optimal predicting function $f(x)$ we define a quadratic error measure

$$E = \sum_i \Big(f(x_i) - y_i \Big)^2. \tag{40}$$

In the particular situation of linear regression the function f is linear, $f(x) = \alpha + \beta x$. (A more flexible f would for example be a neural network.) The optimal parameters α, β are now found as usual by setting $\partial E/\partial\alpha = 0$ and $\partial E/\partial\beta = 0$. Thus,

$$\begin{aligned} 0 &= 2\sum_i \Big(\alpha + \beta x_i - y_i\Big) && (41)\\ 0 &= 2\sum_i x_i\Big(\alpha + \beta x_i - y_i\Big) && (42) \end{aligned}$$

or

$$\begin{aligned} \alpha &= \langle y \rangle - \beta \langle x \rangle, && (43)\\ 0 &= \alpha \langle x \rangle + \beta \langle x^2 \rangle - \langle xy \rangle, && (44) \end{aligned}$$

from which follows

$$\beta = \frac{\langle xy \rangle - \langle x \rangle \langle y \rangle}{\langle x^2 \rangle - \langle x \rangle^2} = \frac{\sigma_{xy}}{\sigma_x^2}. \tag{45}$$

The definition of β being thus the same, Eq. (43) can now be identified with Eq. (37) with α corresponding to m_0, σ_{xy} to C_{ip}, and σ_x^2 to D_p.

4 Nonlinear constraints and spin glasses

Finally, we outline the main idea of how portfolios and spin glasses can be related [3]. This shows that nonlinear constraints can lead to many solutions for the optimal portfolio. Consider a portfolio with futures, for which a margin is required for both sides. Limiting such margins requires an additional constraint

$$\gamma \sum_i |p_i| x_i. \tag{46}$$

Hence, we may define an optimal portfolio as the minimum of

$$\frac{1}{2} \sum_{ij} p_i C_{ij} p_j - \zeta \sum_i p_i m_i - \gamma \sum_i x_i |p_i|. \tag{47}$$

This yields,

$$0 = \sum_j C_{ij} p_j - \zeta m_i - \gamma x_i S_i, \tag{48}$$

i.e.,

$$p_i = \zeta \sum_j C_{ij}^{-1} m_j + \gamma \sum_j C_{ij}^{-1} x_j S_j, \tag{49}$$

where $S_i = \mathrm{sign}(p_i)$. Setting $x_j = 1$, and taking the sign yields

$$S_i = \mathrm{sign} \left[h_i + \sum_j^N J_{ij} S_j \right], \tag{50}$$

with $h_i = \zeta \sum_j C_{ij}^{-1} m_j$ and $J_{ij} = \gamma C_{ij}^{-1} x_j$. This is the equation, for a state which is (locally) stable under the discrete synchronous Hopfield dynamic, $S_i' = \mathrm{sign}\left[h_i + \sum_j J_{ij} S_j\right]$, of a spin glass–like Hamiltonian

$$H = \sum_{ij}^N S_i J_{ij} S_j + \sum_i^N h_i S_i. \tag{51}$$

(For example, in a Hopfield model, $J_{ij} = \sum_a \xi_i^{(a)} \xi_j^{(a)}$, the $\xi_i^{(a)}$ representing patterns to be stored. In an EA-model (Edwards, Anderson) the J_{ij} are Gaussian random variables with distance dependent variance $\sigma_{J_{ij}} = \Delta(|i-j|)$. In a SK-model (Sherrington, Kirkpatrick) the J_{ij} are Gaussian random variables with distance independent variance $\sigma_{J_{ij}} = J^2/N$, For portfolio theory one can choose a covariance $\mathbf{C} = \mathbf{M}^T \mathbf{M} + \mathbf{D}$ built from random matrices $\mathbf{M}$, $\mathbf{D}$.) As one knows that the ground state of spin glasses can be highly degenerated one can expect a similar effect for such portfolios. For a random matrix treatment see [3].

References

[1] J.-P. Bouchaud, M. Potters: *Theory of Financial Risk.* Draft version available from `http://www.science-finance.fr`

[2] E.J. Elton, M.J. Gruber: *Modern Portfolio Theory and Investment Analysis.* John Wiley and Sons, New York, 1995.

[3] S. Galluccio, J.–P. Bouchaud, M. Potters: *Rational Decisions, Random Matrices, and Spin Glasses.* `arXiv:cond-mat/9801209`

[4] H. Markowitz: *Portfolio Selection: Efficient Diversification of Investments.* John Wiley and Sons, 1959; Yale University Press, 1970; Basil Blackwell, 1991.