

# On the depinning of a drop of partially wetting liquid on a rotating cylinder

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We discuss the analogy of the behaviour of films and drops of liquid on a rotating horizontal cylinder on the one hand and substrates with regular one-dimensional wettability patterns on the other hand. On the basis of the similarity between the respective governing long-wave equations, we show that a drop of partially wetting liquid on a rotating cylinder undergoes a depinning transition when the rotation speed is increased. The transition occurs via a sniper bifurcation, as in a recently described scenario for drops depinning on heterogeneous substrates.

**Key words:** bifurcation, interfacial flows (free surface), lubrication theory

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## 1. Introduction

Motivated by the question how much honey can be kept on a breakfast knife by rotating the knife about its long axis, Moffatt (1977) studied the destabilization of a film on the outside of a rotating horizontal cylinder and the subsequent development of regular patterns of azimuthal rings. He also gave a lubrication equation that incorporates gravity and viscosity effects but neglects surface tension and inertia. It is based on a model by Pukhnachev (1977), which includes surface tension. Related questions had been studied earlier by Yih & Kingman (1960), employing a rotating horizontal cylinder which is at the bottom immersed in a liquid bath; i.e. in contrast to Moffatt (1977), the amount of liquid on the cylinder is not an independent control parameter. The related case of a liquid film covering the inner wall of a horizontal rotating cylinder was first studied by Phillips (1960), who discussed experiments performed with water, employing an inviscid model incorporating gravitational and centrifugal forces.

Since the early experimental results, engineers and scientists alike have studied flows of free-surface films of liquid on the inner or outer wall of resting or rotating horizontal cylinders in a number of different settings. The systems studied are of high importance for several coating and printing processes, where rotating cylinders transport the coating material in the form of liquid films. In the present contribution, we will discuss the analogy between film flow and drop motion on a rotating horizontal cylinder on the one hand and on heterogeneous substrates with regular wettability patterns on the other hand. First, however, we will give a short account of the seemingly disconnected fields.

The theoretical analyses given by Yih & Kingman (1960) and Phillips (1960) for the flow on/in a rotating cylinder were based on linear stability considerations

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based on the full hydrodynamic equations for momentum transport, i.e. the Navier–Stokes equations. Their studies were extended later on by e.g. Pedley (1967), who considered inviscid film flow inside and outside of the cylinder in a unified manner, and Ruschak & Scriven (1976), who analysed viscous film flow at small and large Reynolds numbers.

However, most analyses of the nonlinear behaviour are based on a long-wave or lubrication approximation (Oron, Davis & Bankoff 1997) valid for the case in which the thickness of the liquid film  $h(\theta, t)$  is small as compared to the radius of the cylinder  $R$ . The resulting evolution equation for the film thickness profile

$$\partial_t h = -\partial_\theta \{h^3 [\tilde{\alpha} \partial_\theta (\partial_{\theta\theta} h + h) - \tilde{\gamma} \cos(\theta)] + h\} \quad (1.1)$$

was first given by Pukhnachev (1977) and used by Moffatt (1977) (without capillarity effects, i.e.  $\tilde{\alpha} = 0$ ). In the dimensionless equation (1.1),  $\tilde{\alpha} = 3\varepsilon^3 \sigma / R\omega\tilde{\eta}$  and  $\tilde{\gamma} = \varepsilon^2 \rho g R / 3\omega\tilde{\eta}$  are the scaled dimensionless surface tension and gravity parameter, respectively (Hinch & Kelmanson 2003). It is important to note that both depend on the angular velocity of the rotation  $\omega$ , the dynamic viscosity  $\tilde{\eta}$ , the cylinder radius  $R$ , and the ratio  $\varepsilon$  of mean film thickness  $\bar{h}$  and  $R$ . The remaining material constants are surface tension  $\sigma$  and density  $\rho$ , and  $g$  is the gravitational acceleration. The angle  $\theta$  determines the position on the cylinder surface and is measured anticlockwise starting at the horizontal position on the right. The force  $\tilde{\gamma} \cos(\theta)$  corresponds to the component of gravity parallel to the cylinder surface.

The long-wave equation (1.1) including surface tension effects has been studied in detail by e.g. Hinch & Kelmanson (2003), Pukhnachov (2005) and Karabut (2007). Note that for the parameter ranges where (1.1) applies, it is valid without difference for films on the outside and inside of the cylinder (cf. e.g. Ashmore, Hosoi & Stone 2003; Hinch & Kelmanson 2003; Noakes, King & Riley 2006). A preliminary comparison of the long-wave approach to results obtained with the Stokes equation is given by Peterson, Jimack & Kelmanson (2001). Various extensions of (1.1) were developed. See, in particular, Acrivos & Jin (2004) for a discussion of hydrostatic effects. Other higher-order terms related to gravity, inertial and centrifugal effects have been included (Hosoi & Mahadevan 1999; O’Brien 2002; Ashmore *et al.* 2003; Evans, Schwartz & Roy 2004; Benilov & O’Brien 2005; Evans, Schwartz & Roy 2005; Noakes *et al.* 2006; Kelmanson 2009). In particular, Noakes *et al.* (2006) and Kelmanson (2009) present a systematic derivation of several such higher-order models and also give detailed comparisons to a number of previous studies, as e.g. O’Brien (2002), Ashmore *et al.* (2003) and Benilov & O’Brien (2005). The film stability and nonlinear evolution of a film on the outside of a non-isothermal horizontal cylinder was considered without (Reisfeld & Bankoff 1992) and with (Duffy & Wilson 2009) rotation. Most of the aforementioned studies consider two-dimensional situations. The fully three-dimensional problem is studied for the cases without (Weidner, Schwartz & Eres 1997) and with (Evans *et al.* 2005; Noakes, King & Riley 2005) rotation.

The second system that we would like to discuss consists of liquid drops and films on heterogeneous solid substrates as studied experimentally, e.g. by Schwartz & Garoff (1985) and Quéré, Azzopardi & Delattre (1998). Although in principle the heterogeneity may result from substrate topography or chemical heterogeneities, our focus here is on the case of a smooth flat substrate with chemical heterogeneities, i.e. we assume that the substrate wettability depends on position. Horizontal heterogeneous substrates are often considered in connection with micro-patterning of soft matter films via thin film dewetting (Rockford *et al.* 1999; Bauer & Dietrich 2000; Konnur, Kargupta & Sharma 2000; Bruschi *et al.* 2002). Another application relates to

free-surface liquid channels in microfluidics (Gau *et al.* 1999). Often, one employs thin film evolution equations obtained through a long-wave approximation to study the dynamics of dewetting for an initially flat film or to analyse steady drop and ridge solutions and their stability on such patterned substrates (see e.g. Kargupta & Sharma 2003; Thiele *et al.* 2003). Static structures may also be studied using a variational approach (Brandon & Marmur 1996; Lenz & Lipowsky 2000).

More recently, a model system was introduced to study the dynamics of drops on heterogeneous inclined substrates. The models also apply to such drops under other driving forces along the substrate (Thiele & Knobloch 2006*b*). The corresponding dimensionless evolution equation for the one-dimensional film thickness profile (i.e. describing a two-dimensional drop) is of the form

$$\partial_t h = -\partial_x \{h^3 [\partial_x (\partial_{xx} h + \Pi(h, x))] + \tilde{\mu}\}, \quad (1.2)$$

where  $\tilde{\mu}(h)$  represents the lateral driving force ( $\mu h^3$  in the case of an inclined substrate, where  $\mu$  is the inclination angle). The position-dependent disjoining pressure  $\Pi(h, x)$  models the heterogeneous wettability of the substrate. One notices at once that (1.2) and (1.1) are rather similar. By identifying  $x = \theta$ ,  $\tilde{\mu}(h) = h$  and  $\Pi(h, x) = \tilde{\alpha}h - \tilde{\gamma} \sin(x)$ , (1.1) may for  $\tilde{\alpha} = 1$  be seen as a special case of (1.2). Here, we use the analogy to study drops or films on a rotating cylinder. In particular, we show that drops on a rotating cylinder may show rather involved depinning dynamics analogous to the behaviour of drops on heterogeneous substrates (Thiele & Knobloch 2006*a,b*; Beltrame, Hänggi & Thiele 2009).

To illustrate this behaviour, we choose the case of drops of a partially wetting liquid on a substrate with a periodic array of localized hydrophobic wettability defects (Thiele & Knobloch 2006*b*). On a horizontal substrate, one finds steady drops that are positioned exactly between adjacent defects, i.e. the drops stay on the most wettable part. At a small lateral driving force  $\mu$ , the  $x \rightarrow -x$  symmetry of the system is broken. However, the drops do not slide continuously along the substrate as they do in the case of a substrate without defects (Thiele *et al.* 2001). Their advancing contact line is blocked by the next hydrophobic defect, i.e. one finds steady pinned drops at an upstream position close to the defect. As the driving force is increased, the drops are pressed further against the defect. As a result, the steady drops steepen. At a finite critical value  $\mu_c$ , the driving force allows the drop to overcome the adverse wettability gradient and the drop depins from the defect and begins to slide. In a large part of the parameter space spanned by the wettability contrast and the drop size, the point of depinning at  $\mu_c$  corresponds to a saddle-node infinite period (sniper) bifurcation. Beyond  $\mu_c$ , the drops do not slide continuously down the incline. They rather perform a stick-slip motion from defect to defect (Thiele & Knobloch 2006*a,b*). However, in another region of the parameter space, the depinning can occur via a Hopf bifurcation, i.e. with a finite frequency. In particular, this happens in the case of rather thick wetting layers.

The aim of the present contribution is to elucidate under which conditions a similarly intricate ‘depinning’ behaviour can be found for a pendant drop underneath a horizontal cylinder when increasing the speed of rotation. In §2, we will discuss the scaling and introduce a long-wave model for a film of partially wetting liquid on a rotating cylinder. It allows (even without gravitation) for the coexistence of drops and a wetting layer. We discuss several physical situations in which it can be applied. Next, §3 presents steady-state solutions depending on the non-dimensional rotation velocity for several thicknesses of the wetting layer and discusses the depinning scenario in

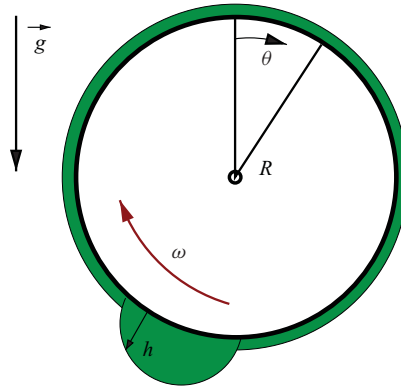


FIGURE 1. (Colour online at <http://cambridge.journals.org/film>) Sketch of a drop of partially wetting liquid coexisting with a thick wetting layer on a rotating cylinder.

the case of the partially wetting liquid. Finally, we conclude and give an outlook in §4.

## 2. Model

Before we introduce the long-wave model, we discuss our scales. In the non-dimensionalization used in most works studying (1.1), the angular velocity of the cylinder rotation  $\omega$  is not reflected in a single dimensionless parameter but both dimensionless parameters are inversely proportional to  $\omega$ . Furthermore, the time scale depends on  $\omega$  (Pukhnachev 1977; Hinch & Kelmanson 2003). This and a further rescaling of the equation for steady drop and film states, which absorbs the flow rate into both dimensionless parameters and into the thickness scale, implies that the existence of two steady solutions for identical parameters (thin capillary film and pendant drop with through-flow), shown analytically by Pukhnachov (2005) and numerically by Karabut (2007), actually refers to solutions of different liquid volumes.

Such a scaling is not convenient for our study as we would like to investigate the change in system behaviour for fixed liquid volume when increasing the rotation speed of the cylinder from zero, i.e.  $\omega$  should only enter a single dimensionless parameter that will be our main control parameter. A scaling fit for our purpose is the one based on the scales of the gravity-driven drainage flow employed by Evans *et al.* (2004).

Note also that in particular Karabut (2007) and to a lesser extent Weidner *et al.* (1997) represent some of their solutions in a way that misrepresents their findings and, in general, the concept of an equation in long-wave approximation. Whenever one shows the solutions as a thickness profile on a cylinder, one has to pick a particular radius  $R$  and smallness ratio  $\epsilon$  for the representation (cf. figures 3, 5 and 9). This is a rather arbitrary choice as the equation in long-wave approximation is strictly valid only in the limit  $\epsilon \rightarrow 0$ , i.e. it is always only an approximation to a real physical situation where  $\epsilon$  is finite.

In the present work, we focus on solutions without axial variation, i.e. we consider the two-dimensional physical situation shown in figure 1. Following Evans *et al.* (2004), we use  $\bar{h}$ ,  $R$ ,  $\rho g \bar{h}^2 / \eta$ ,  $\epsilon \rho g \bar{h}^2 / \eta$ ,  $3\eta / \epsilon \bar{h} \rho g$  and  $\rho g \bar{h}$  to scale the coordinate  $z$  orthogonal to the cylinder surface, the coordinate  $x = \theta R$  along the surface, the two velocity components  $v_x$  and  $v_z$ , time and pressure, respectively. The ratio of the  $z$ - and  $x$ -scales is the smallness parameter  $\epsilon = \bar{h} / R$ , i.e. the film height is scaled by  $\bar{h} = \epsilon R$ .

Here, the angle  $\theta$  is defined in the clockwise direction starting at the upper vertical position, i.e. a pendant droplet underneath the resting horizontal cylinder has its centre of mass at  $\theta = \pi$ .

The evolution equation is derived from the Navier–Stokes equations with no-slip boundary conditions at the (rotating) cylinder, and tangent and normal force equilibria at the free surface using the long-wave approximation (Pukhnachev 1977; Reisfeld & Bankoff 1992; Oron *et al.* 1997; Evans *et al.* 2004). We furthermore assume that the ratio  $\bar{h}/R$  and the physical equilibrium contact angle are both  $O(\varepsilon)$  (cf. §4). This gives

$$\partial_\tau h = -\partial_\theta \{h^3 \partial_\theta [Bo^{-1} (\partial_{\theta\theta} h + h) - \cos(\theta) + \tilde{\Pi}(h)] + \tilde{\Omega} h\}, \quad (2.1)$$

where  $\tau$  is the non-dimensional time, and

$$Bo = \frac{R^3 \rho g}{\bar{h} \sigma} \quad \text{and} \quad \tilde{\Omega} = \frac{\eta \omega R}{\rho \bar{h}^2 g} \geq 0 \quad (2.2)$$

are an effective Bond number and a rotation number for a clockwise rotation, respectively. The latter corresponds to the ratio of the rotation velocity at the cylinder surface and the drainage velocity. The function  $\tilde{\Pi}(h)$  is a non-dimensional disjoining pressure that accounts for the wettability of the liquid film on the cylinder surface (de Gennes 1985; Israelachvili 1992; Bonn *et al.* 2009).

To make the analogy with the depinning droplet on a heterogeneous substrate more obvious, we introduce another time scale  $t = \tau/Bo$ , disjoining pressure  $\Pi = Bo \tilde{\Pi}$  and the rotation number  $\Omega = Bo \tilde{\Omega}$  and obtain

$$\partial_t h = -\partial_\theta \{h^3 \partial_\theta [\partial_{\theta\theta} h + h - Bo \cos(\theta) + \Pi(h)] + \Omega h\}, \quad (2.3)$$

with

$$\Omega = \frac{\eta \omega R}{\varepsilon^3 \sigma}. \quad (2.4)$$

Defining the derivative

$$\partial_h f = -\Pi(h) - h + Bo \cos(\theta) \quad (2.5)$$

of a ‘local free energy’  $f(h, \theta)$  and identifying  $\tilde{\mu} = \Omega h$ , (2.3) takes the form of (1.2) and the Bond number takes the role of a ‘heterogeneity strength’. Note that the scaling used is well adapted for discussing independently the influences of rotation and gravity in the presence of capillarity. However, it does not allow for a study of the limit of vanishing surface tension.

As disjoining pressure we choose here the combination of a long- and a short-range power law  $\Pi(h) = Ha/h^3 (1 - b/h^3)$ , where  $Ha$  is a dimensionless Hamacker constant. For  $b > 0$ , one has a precursor film or wetting layer thickness  $h_0 = b^{1/3}$  (Pismen 2001; Pismen & Thiele 2006; Thiele 2010). This is valid only for  $Ha < 0$ , i.e. when the long-range interaction is destabilizing. This corresponds to the case of a partially wetting liquid. For  $Ha > 0$ , a rather thick wetting layer can be stabilized and the  $b/h^6$  term is no longer relevant and could just as well be dropped.

Note that  $\partial_h f$  may contain other terms besides those in (2.5). For a heated or cooled cylinder, a term  $(3/2) Bo Bi Ma \log[h/(1 + Bi h)] + 1/(1 + Bi h)$  would enter where  $Bi$  is the Biot number and  $Ma$  is an effective Marangoni number (Oron & Rosenau 1992; Reisfeld & Bankoff 1992; Thiele & Knobloch 2004). If the rotating cylinder forms the inner electrode of a cylindrical capacitor, a term  $V_0/[h + (d - h)\varepsilon]^2$  has to be included for a dielectric liquid and DC voltage of non-dimensional strength  $V_0$  ( $\varepsilon$  is the electric permittivity and  $d$  is the distance between the two cylinders (see the Appendix of

Thiele & Knobloch 2006*b* and John, Hänggi & Thiele 2008). The similarity in the behaviour of depinning drops under the influence of different physical effects when the effective pressure terms ‘look similar’ is discussed by Thiele & Knobloch (2006*b*).

Although we do not consider such other physical effects explicitly, we will explore the effect of corresponding thick wetting layers by adapting the parameters of our disjoining pressure accordingly. A thick wetting layer that coexists with droplets might, for instance, result from the interplay of a destabilizing thermal or electrical effect and a stabilizing long-range van der Waals interaction. We obtain adequate values of  $Ha$  and  $b$  for the present  $\Pi(h)$  by relating them to the (non-dimensional) thickness of the wetting layer  $h_0$  and a static macroscopic contact angle  $\beta_0$  by (see e.g. de Gennes 1985)

$$b = h_0^3 \quad \text{and} \quad Ha = -\frac{5}{3}\beta_0^2 h_0^2. \quad (2.6)$$

Note that  $\beta_0$  is the angle in long-wave scaling, i.e. a small physical equilibrium contact angle  $\beta_{eq} = \varepsilon\beta_0$  corresponds to a long-wave contact angle  $\beta_0$  of  $O(1)$ . In the limit of a non-rotating cylinder (i.e.  $\Omega = 0$ ) and assuming that the disjoining pressure corresponds to complete wetting ( $Ha > 0$ ,  $1/h^6$  term may be dropped), (2.3) corresponds to that given by Reisfeld & Bankoff (1992) for the isothermal case. However, they first use a viscous scaling, which they transform in the second step – a rescaling of time and therefore velocities with the Galileo number – into the ‘drainage scaling’ used here. Equation (2.3) also corresponds to that of Hinch & Kelmanson (2003) when the disjoining pressure is added to their equation.

To analyse the system behaviour, we next determine the steady-state solutions that correspond e.g. to pendant droplets. The behaviour beyond the depinning threshold is analysed using time-stepping algorithms. As we restrict ourselves to the two-dimensional physical situation, an explicit scheme for stiff equations suffices for the latter. The steady-state solutions are obtained using the continuation techniques of the package AUTO (Doedel *et al.* 1997). The steady and time-periodic solutions are characterized by their  $L^2$ -norm  $\|\delta h\| \equiv \sqrt{(1/2\pi) \int_0^{2\pi} (h(\theta) - 1)^2 d\theta}$  and the time-averaged  $L^2$ -norm  $\|\delta h\| \equiv \sqrt{(1/2\pi T) \int_0^T \int_0^{2\pi} (h(\theta) - 1)^2 d\theta dt}$ , respectively. Here  $T$  is the time period.

Note, finally, that (2.3) is the result of our choice for the relative order of magnitude of the involved dimensionless numbers that is based on the choice of physical effects that shall be discussed (cf. Oron *et al.* 1997). Once this is accepted, (2.3) is asymptotically correct to  $O(1)$ . Its form as a conservation law implies that  $\int_0^{2\pi} h d\theta = 2\pi$ . To  $O(1)$ , this corresponds to mass conservation. For a discussion of higher-order corrections to this picture, see Kelmanson (2009).

### 3. Results

#### 3.1. Parameters

We base our estimate for realistic Bond and rotation numbers on two liquids typically studied in the literature: case (i) is water at 25°C as in Reisfeld & Bankoff (1992) and case (ii) is silicone oil with  $\sigma = 0.021 \text{ N m}^{-1}$ ,  $\eta = 1 \text{ kg (m s)}^{-1}$  and  $\rho = 1200 \text{ kg m}^{-3}$  as in Evans *et al.* (2004). As experimentally feasible configurations, we assume angular rotation velocities  $\omega = 0.1\text{--}10 \text{ s}^{-1}$ , cylinders of radii  $R = 10^{-3}\text{--}10^{-2} \text{ m}$  and smallness ratios  $\varepsilon = 0.01\text{--}0.1$ . It is only for the purpose of illustration of steady droplets on a cylinder that we later use  $\varepsilon = 0.1$  in selected figures.

For  $\omega = 1 \text{ s}^{-1}$  and a smallness ratio  $\varepsilon = 0.1$ , we find for a cylinder of radius  $R = 10^{-2} \text{ m}$  for case (i)  $Bo = 136$  and  $\Omega = 0.14$ , whereas case (ii) gives  $Bo = 561$  and  $\Omega = 476$ . For a smaller cylinder of radius  $R = 10^{-3} \text{ m}$ , one has (i)  $Bo = 1.36$  and  $\Omega = 0.014$  and (ii)  $Bo = 5.61$  and  $\Omega = 47.6$ . Decreasing the smallness ratio to  $\varepsilon = 10^{-2}$  increases all Bond numbers by a factor 10 and all rotation numbers by a factor  $10^3$ . Focusing on cylinder diameters in the millimetre range, we mainly investigate  $Bo \leq 10$ . Note that experiments may also be performed with larger cylinders. To keep the Bond number in the interesting range, one could decrease the relevant density by replacing the ambient gas by a second (immiscible) liquid. A feasible experimental set-up could be a horizontal Taylor–Couette apparatus with a rotating inner cylinder. The theoretical framework would also need to be amended – replacing the one-layer theory (2.3) by a closed two-layer model similar to Merkt *et al.* (2005).

Note that silicone oils may have viscosities two orders of magnitude smaller or larger than the chosen value, and one is able to vary the velocity of rotation in a wide range. This implies that there is some flexibility in the choice of  $\Omega$ . We will find that for  $Bo = O(1)$ , the interesting range for the rotation number is  $\Omega = O(1)$ . For instance, with  $Bo = 1$  one finds that depinning occurs at  $\Omega = 1.68$ .

As the present work aims at establishing the qualitative correspondence between the behaviour of a depinning drop on a heterogeneous substrate and that of a droplet on a rotating cylinder, we restrict present results to the partially wetting case (choosing  $h_0 = 0.1$  and  $\beta_0 = 2$ ) and the completely wetting case ( $\beta_0 = 0$ ). For comparison, selected results are also shown for the contact angle  $\beta_0 = 1$ .

Note that here the scaling and therefore the discussion of the smallness parameter  $\varepsilon$  is based on  $\varepsilon = \bar{h}/R$ , i.e. we use the ratio of mean film thickness and cylinder radius. In the case of pendant drop solutions, one also has to keep the maximal film height  $h_{max}$  in mind when discussing  $\varepsilon$ , as the scaling becomes questionable when  $h_{max} \gg \bar{h}$ . This is, however, normally not the case. In the present work, the ratio  $h_{max}/\bar{h}$  is always below 3. Although an inspection of the figures in Karabut (2007) shows a ratio  $h_{max}/\bar{h}$  that is not much larger, one notices that there  $\varepsilon$  itself (based on  $\bar{h}$ ) seems to be larger than one (see remark above (2)). The same applies to figure 3 of Weidner *et al.* (1997). No such problem exists in the region where only a thin wetting layer of thickness  $h_{min}$  covers the cylinder even if  $h_{min} \ll \bar{h}$ . Actually, the long-wave approximation gets better there. In an approach based on matched asymptotics, one could further simplify the governing equation in this region (cf. Benilov, Benilov & Kopteva 2008). However, such an approach is not taken here as it is of limited use when considering parameter regions that show qualitative changes in solution behaviour, e.g. close to the sniper bifurcation discussed below.

### 3.2. Partially wetting case

First, we consider the case of a horizontal cylinder without rotation ( $\Omega = 0$ ) and determine steady drop and film solutions. Employing the Bond number as control parameter, we obtain families of steady-state solutions for two selected values of the equilibrium contact angle  $\beta_0$ . Inspecting figure 2 one notices that above a critical value of the Bond number  $Bo_c$  there exists only a single solution. It corresponds to a symmetric pendant drop located underneath the cylinder. An example for  $Bo = 1$  and  $\beta_0 = 2.0$  is given by the solid line in figure 3. However, two additional steady solutions exist below  $Bo_c$ . One of them corresponds to an unstable symmetric drop sitting on top of the cylinder (dotted line in figure 3). The other one corresponds to an unstable solution that has two minima – a deep minimum underneath the cylinder and a shallow one on top of it (dashed line in figure 3).

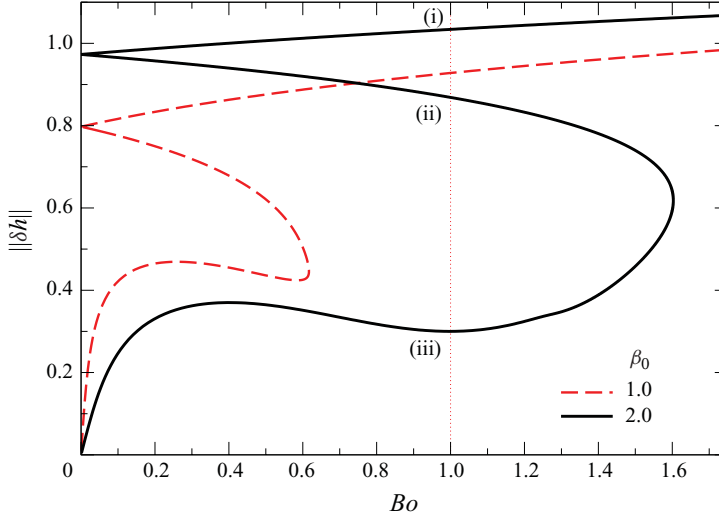


FIGURE 2. (Colour online) Solutions on a horizontal cylinder without rotation ( $\Omega = 0$ ) are characterized by their  $L^2$ -norm as a function of the Bond number for two different equilibrium contact angles  $\beta_0$  and  $h_0 = 0.1$ ,  $\bar{h} = 1.0$ . Symbols (i), (ii), and (iii) mark solutions for  $Bo = 1.0$  (indicated by the vertical dotted line) that are shown in figure 3 and are discussed in the main text.

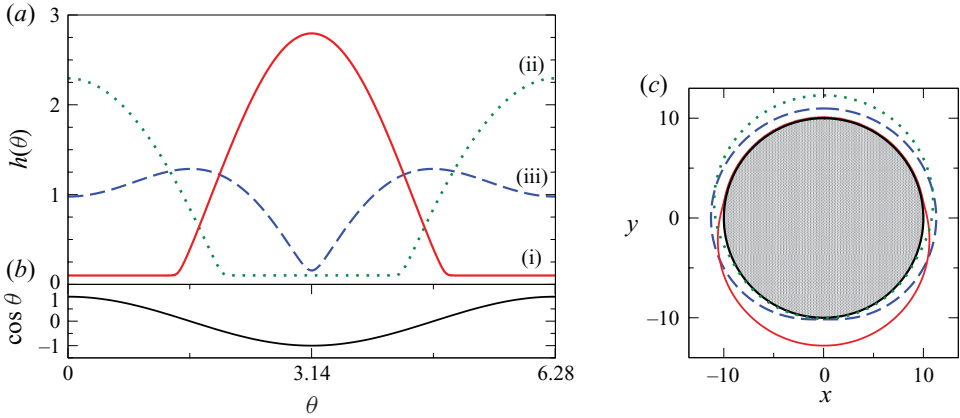


FIGURE 3. (Colour online) Steady film thickness profiles on a horizontal cylinder without rotation ( $\Omega = 0$ ) for  $Bo = 1$  and  $\beta_0 = 2.0$  (corresponding to points (i)–(iii) in figure 2). The remaining parameters are the same as in figure 2. (a) Profiles as  $h(\theta)$ , (b) the space-dependent part of  $\partial_h f$  (2.5) and (c) the profiles on a cylinder. Note that for illustration purposes, we have (rather arbitrarily) assumed a radius  $R = 10$ , i.e.  $\varepsilon = 0.1$  (cf. discussion in § 3.1). Line styles in (c) correspond to those in (a). The cylinder surface is represented by the solid line.

We find that  $Bo_c$  becomes smaller with decreasing contact angle  $\beta_0$ , i.e. the range of  $Bo$  where steady structures beside the pendant drop exist becomes smaller. The completely wetting case ( $\beta_0 = 0$ ) is qualitatively different and is discussed in § 3.3. By increasing  $\beta_0$  above  $\beta_0 = 2$ , the critical value  $Bo_c$  strongly increases, e.g. for  $\beta_0 = 10$  one finds  $Bo_c = 43.3$ .

Note that the pendant drop underneath the cylinder (drop on top of the cylinder) corresponds on a horizontal heterogeneous substrate to the drop on the more (less) wettable region as discussed by Thiele *et al.* (2003). The limit of zero Bond number is analogous to the horizontal homogeneous substrate.



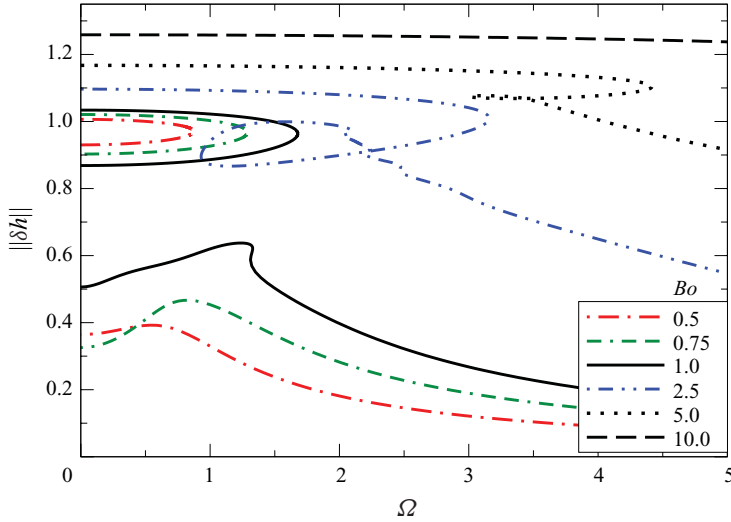


FIGURE 4. (Colour online) Solutions on a horizontal rotating cylinder as a function of the rotation number,  $\Omega$ , for various Bond numbers,  $Bo$ , as given in the legend and contact angle  $\beta_0 = 2.0$ . The drop and film profiles are characterized by their  $L^2$ -norm. The remaining parameters are  $h_0 = 0.1$  and  $\bar{h} = 1.0$ .

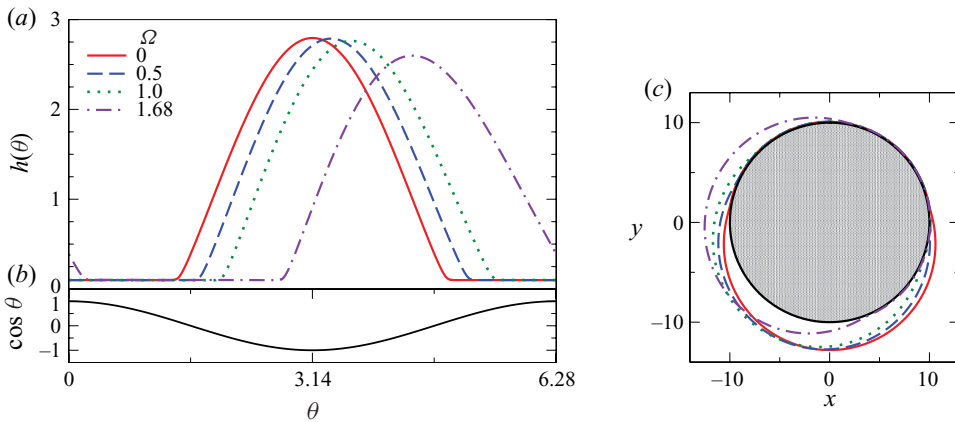


FIGURE 5. (Colour online) Steady film thickness profiles on a rotating horizontal cylinder for  $Bo = 1$  and  $\beta_0 = 2.0$  for rotation numbers  $\Omega$  as given in the legend. All solutions shown are situated on the stable upper branch in figure 4. The remaining parameters are the same as in figure 4. (a) The profiles as  $h(\theta)$ , (b) the space-dependent part of  $\partial_h f$  (2.5) and (c) the profiles on a cylinder assuming a radius  $R = 10$ . Line styles in (c) correspond to those in (a).

Next, we increase the dimensionless angular velocity  $\Omega$  for several selected Bond numbers and determine the steady-state solutions (figure 4). Focusing first on  $Bo = 1$ , we recognize at  $\Omega = 0$  the three solutions indicated by symbols (i), (ii), and (iii) on the vertical dotted line in figure 2. The curve of the largest norm corresponds to the pendant drop. As  $\Omega$  increases from zero, its norm and amplitude decrease slightly, whereas its centre of mass moves towards larger  $\theta$ , that is, it is ‘dragged along’ by the rotation without changing its shape much. This can be well appreciated in figure 5, which gives selected profiles for the branch of steady pendant drop solutions. The branch terminates in a saddle-node bifurcation at  $\Omega_{sn} = 1.68$ , where it annihilates with one of the unstable branches. The third branch, i.e. the one of the lowest norm,

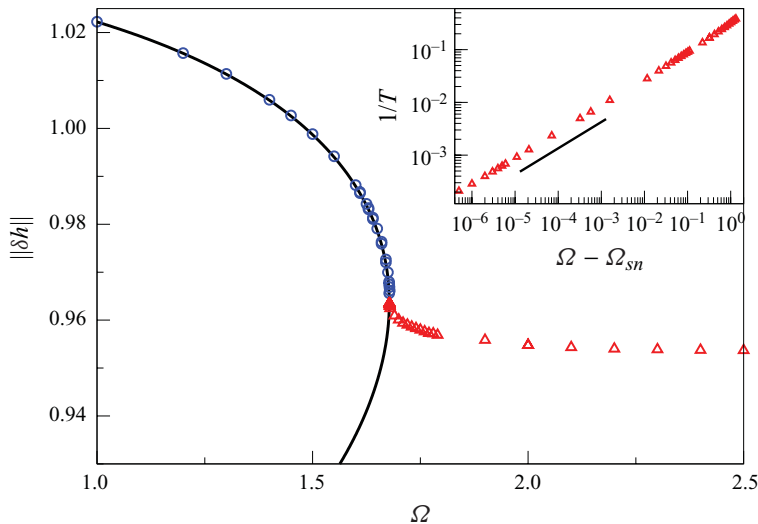


FIGURE 6. (Colour online) Bifurcation diagram for the depinning transition for a drop on a rotating cylinder with  $Bo = 1$  and  $\beta_0 = 2.0$ . Shown are the  $L^2$ -norm for the branch of steady solutions as obtained by continuation (solid line), selected steady solutions as obtained by direct integration in time (circles) and the time-averaged  $L^2$ -norm for the unsteady solutions beyond the sniper depinning bifurcation (triangles). The inset gives, for the latter, a log–log plot of the dependence of the inverse temporal period on the distance from the bifurcation  $\Omega - \Omega_{sn}$ . The solid line indicates an exponent of  $1/2$ .

shows for increasing  $\Omega$  an increase in the norm, then it undergoes two saddle-node bifurcations at  $\Omega = 1.313$  and  $\Omega = 1.329$ . At larger  $\Omega$ , the norm decreases again. Note that the two saddle-node bifurcations are a first sign of rather complicated reconnections which occur when one increases  $Bo$  from 1.0 to 2.5. The reconnections involve further (unstable) branches that are not shown in figure 4 and result in the loop structure observed for  $Bo = 2.5$  at about  $\Omega = 1.5$ . A further indication for the existence of additional branches is the observation that the value of the norm at  $\Omega = 0$  for  $Bo = 1.0$  in figure 4 does not agree with the lowest norm at  $Bo = 1.0$  in figure 2. As the reconnections are not relevant for the depinning transition, we will not consider them further.

At  $Bo$  smaller than 1, the behaviour is qualitatively the same as described for  $Bo = 1$ . However, the critical  $\Omega_{sn}$  becomes smaller with decreasing  $Bo$ . Above, we have discussed that the behaviour at  $\Omega = 0$  changes qualitatively when increasing  $Bo$  beyond  $Bo_c$  (figure 2). Although this also affects the behaviour at small  $\Omega$ , it has less influence on the behaviour at larger  $\Omega$  (figure 4). In particular, the saddle node at  $\Omega_{sn}$ , where the pendant drop solution ceases to exist, persists for larger Bond number as does the related dynamic behaviour (see below). Note that for  $Bo = 10$ , the saddle node is at  $\Omega_{sn} \approx 6.2$ , outside the range of figure 4.

An interesting resulting question is what happens to a stable obliquely pendant droplet at  $\Omega < \Omega_{sn}$ , when the rotation velocity is further increased such that it is slightly larger than  $\Omega_{sn}$ ? Does the solution approach the remaining steady solution? Simulations of (2.3) show that this is not the case. For  $\Omega > \Omega_{sn}$ , the droplet moves in a non-stationary way with the rotating cylinder and corresponds to a space- and time-periodic solution. The resulting bifurcation diagram for  $Bo = 1$  is given in figure 6. This shows that the ‘new’ branch of space- and time-periodic solutions (characterized by its time-averaged norm) emerges from the saddle-node bifurcation

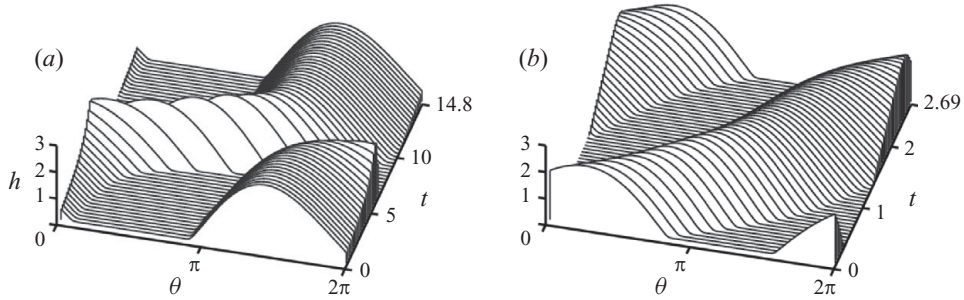


FIGURE 7. Space–time plots illustrating the time evolution of co-rotating drops beyond depinning via a sniper bifurcation (at  $\Omega > \Omega_{sn}$ ). One period in space and time is shown (a) close to depinning at rotation number  $\Omega = 1.8$  with a temporal period of  $T = 14.8$  and (b) far from depinning at  $\Omega = 3.0$  with  $T = 2.7$ . Here  $Bo = 1$  and  $\beta_0 = 2.0$ .

of the steady solutions at  $\Omega_{sn}$ . The behaviour close to and far from the saddle-node bifurcation is illustrated in the space–time plots of figure 7. Figure 7(b) shows that far away from the bifurcation the drop moves continuously, with its velocity and shape varying smoothly. It moves fastest when its maximum passes  $\theta \approx \pi/2$  (velocity  $\approx 4.6$ ) and slows down when passing  $\theta \approx 3\pi/2$  (velocity  $\approx 1.2$ ), i.e. respectively, when gravity most strongly supports and hinders the motion driven by the cylinder rotation. The ratio of largest to smallest velocity is about 4:1. When approaching the bifurcation from above, the droplet still moves with the rotation, but the time scales of the slow and fast phases become very different. For instance, at  $\Omega = 1.8$  (figure 7a), the drop moves fastest when passing  $\theta \approx \pi/2$  (velocity  $\approx 3.3$ ). However, when the drop is situated at about  $\theta = 3\pi/2$ , it barely moves (velocity  $\approx 0.1$ ), resulting in a velocity ratio of 33:1 between the fastest and slowest phases. The resulting overall behaviour strongly resembles the stick-slip motion discussed in the context of contact line motion on heterogeneous substrates (Thiele & Knobloch 2006b). Note that it also resembles the so-called sloshing modes found for a partially liquid-filled rotating cylinder (Thoroddsen & Mahadevan 1997; Hosoi & Mahadevan 1999).

The ratio of the velocities diverges when approaching the bifurcation point. In consequence, the frequency  $1/T$  of the periodic drop motion becomes zero. As shown in the inset of figure 6, the dependence corresponds to a power law  $1/T \sim (\Omega - \Omega_{sn})^{1/2}$ . This indicates that the bifurcation at  $\Omega_{sn}$  is actually a saddle-node-infinite-period bifurcation (or ‘sniper’ for short; see Strogatz 1994 and discussion by Thiele & Knobloch 2006b).

The behaviour described is quite generic for the partially wetting case, i.e. it is found in a wide range of parameters, in particular, Bond number, contact angle and wetting layer thickness. However, the behaviour changes dramatically when we strongly decrease the contact angle, i.e. when approaching the completely wetting case.

### 3.3. The completely wetting case

After having discussed the intricate depinning behaviour in the partially wetting case, to emphasize the contrast, we consider briefly the completely wetting case ( $\beta_0 = 0$ ). Note that the physical setting is then identical to that employed in most studies of the classic Moffatt problem (Moffatt 1977; Pukhnachev 1977). However, the parametrization used here is different and results in (2.3), with  $Ha = 0$ . We analyse the system behaviour as above by studying the steady-state solutions without and with rotation.

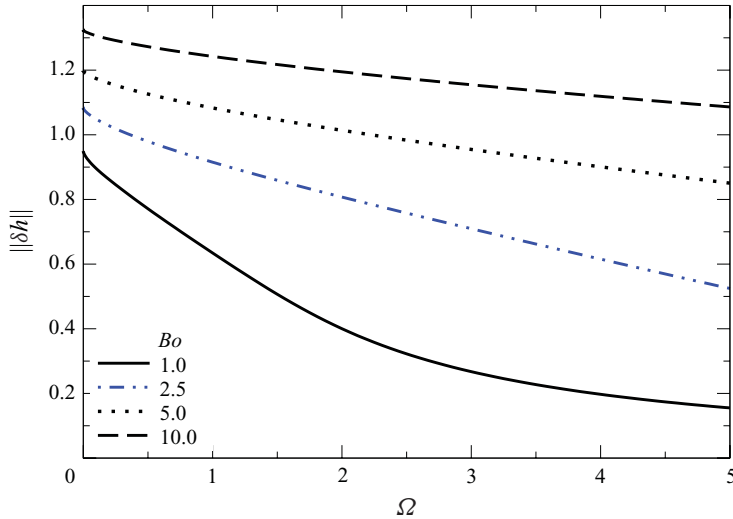


FIGURE 8. (Colour online) Solutions for a film of completely wetting fluid ( $\beta_0 = 0$ ) on a horizontal rotating cylinder as a function of the rotation number  $\Omega$  for various  $Bo$  as given in the legend. The solutions are characterized by their  $L^2$ -norm. The remaining parameter is  $\bar{h} = 1.0$ .

In the case without rotation, the first difference one notices is that for a fixed Bond number, only one solution exists. It corresponds to the pendant drop solution. With increasing  $Bo$ , i.e. with increasing importance of gravity, its amplitude monotonically increases. Fixing  $Bo$  and increasing the rotation velocity from zero, one finds a monotonic decrease of the norm (figure 8) and height of the drop. Its maximum moves towards larger angular position  $\theta$ , i.e. the liquid is dragged along with the rotation. However, in the wetting case, there is no force besides gravity that can favour drops as compared to a flat film. As gravity acts downwards, it only favours pendant drops. Therefore, the drop does not survive as a ‘coherent structure’ when dragged upwards with the rotation: it is flattened and smeared out. Typical profiles are given in figure 9. Note that we do not show the exact case without rotation, but choose  $\Omega = 10^{-10}$ . In this way a dynamically created wetting layer still exists, which is numerically advantageous (cf. Thiele & Knobloch 2004).

When increasing  $\Omega$  further than shown in figure 8, the thickness profiles approach a flat film. For instance, for  $Bo = 1$ , the norm decreases below  $10^{-2}$  at  $\Omega \approx 70$ .

#### 4. Conclusion

Based on the observation that the equations in long-wave approximation that govern film flow and drop motion (i) on or in a rotating cylinder and (ii) on a heterogeneous substrate are rather similar, we have explored whether the analogy can be exploited, i.e. whether it allows results obtained for one system to be transferred to the other system. In particular, we have found that indeed for drops on a rotating cylinder there exists a counterpart of the rather involved depinning dynamics described recently for drops on heterogeneous substrates.

To study the effect, we have introduced an alternative scaling. This has been necessary because the commonly used scaling contains the angular velocity of the cylinder rotation in both dimensionless parameters and in the time scale. Together with a further re-scaling of the steady-state equation involving the flow rate (Pukhnachev

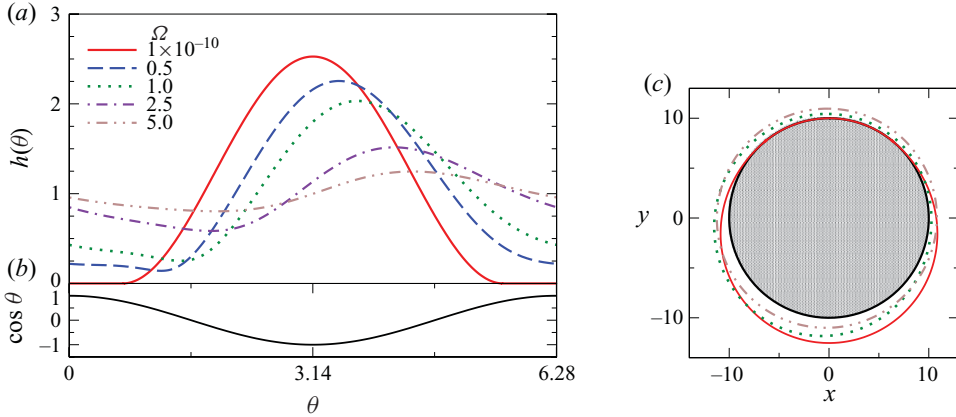


FIGURE 9. (Colour online) Steady film thickness profiles for the case of complete wetting with  $Bo = 1$  for rotation numbers  $\Omega$  as given in the legend. The remaining parameters are the same as in figure 8. (a) The profiles as  $h(\theta)$ , (b) the space-dependent part of  $\partial_h f$  (2.5) and (c) illustrates the profiles on a cylinder assuming a radius  $R = 10$ . Line styles in (c) correspond to those in (a).

1977; Karabut 2007), this does not allow for an investigation of either the system behaviour when increasing the rotation speed of the cylinder from zero or the existence of multiple solutions for fixed rotation speed and liquid volume. The scaling employed here does allow for such studies as the rotation speed only enters a single dimensionless parameter. Furthermore, it allows us to discuss the analogy between the two systems of interest in a rather natural way. In particular, downward gravitation and rotation speed for the drop on the rotating cylinder correspond to the heterogeneous wettability and lateral driving force, respectively, for drops on heterogeneous substrates.

Guided by this analogy, we have studied the behaviour of drops of partially wetting liquid on the rotating cylinder. As a result, it has been shown how ‘switching on’ gravity (increasing the Bond number) without rotation changes the solution behaviour dramatically as the ‘gravitational heterogeneity’ along the cylinder surface effectively suppresses multidrop solutions until only the single pendant droplet underneath the cylinder survives. We have, furthermore, found that increasing the rotation from zero to small values, the drop is dragged along to a stable equilibrium position where downward gravity and upward drag compensate. At a critical speed, however, the retaining downward force is too small and the drop is dragged above the left horizontal position. It then continuously moves with the rotating cylinder in a non-constant manner. Close to the transition, the drop shows stick-slip motion. An analysis of the time-dependent behaviour has shown that the frequency related to the periodic motion becomes zero at the threshold following a power law with power  $1/2$ . This and the related bifurcation diagram show that the observed transition corresponds to depinning via a sniper bifurcation. The behaviour found is rather generic for the case of partial wetting. Although here we have presented results for particular parameter values, it can be observed in a wide range of parameters, in particular, Bond number, equilibrium contact angle and wetting layer thickness. However, we have also found that the behaviour changes dramatically when strongly decreasing the equilibrium contact angle, i.e. when approaching the completely wetting case that is normally studied in the literature. When increasing the velocity of rotation in this case, we have

seen a transition from pendant drops underneath the cylinder to a nearly uniform film around the cylinder, as in Evans *et al.* (2004).

The present results indicate that it may be very fruitful to further explore the analogy of films/drops on rotating cylinders and heterogeneous substrates. We expect that the recent exploration of the three-dimensional case for depinning drops (Beltrame *et al.* 2009) and the various transitions between ridge, drop and rivulet states (Beltrame *et al.* 2010), based on tools for the continuation of steady and stationary solutions of the two-dimensional thin film equation (physically three-dimensional system) (Beltrame & Thiele 2010), is also relevant for the case of a rotating cylinder. A future study could elucidate the relation between the formation of azimuthal rings, pendant ridges and sets of drops.

The present circular cylinder is an analogy to a sinusoidal wettability profile. Cylinders with other cross-sections, for instance, an elliptical cylinder (Hunt 2008) would correspond to more localized defects of the corresponding heterogeneous substrate. However, the analogue system turns out to be more complicated as a rotating non-circular cylinder introduces the aspect of a time-periodic non-harmonic forcing parallel to the substrate into the heterogeneous substrate system.

Equation (2.3) is derived under two conditions: (i) the mean (and also the maximal) film thickness has to be small as compared to the radius of the cylinder and (ii) the local surface slope (e.g. the physical equilibrium contact angle) has to be small. Here, we have assumed that the two related smallness parameters are of the same order of magnitude. In a next step, one may introduce two different smallness parameters and discuss a number of distinct limits in dependence of their ratio. Such a systematic asymptotic study would also permit to discuss the influence of other effects on the depinning behaviour, such as the effects of the hydrostatic pressure (Ashmore *et al.* 2003; Acrivos & Jin 2004), centrifugal forces (Evans *et al.* 2004) and inertia (Hosoi & Mahadevan 1999; Kelmanson 2009).

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